

Limits, Infinity, and Asymptotes

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Overview

The concept of a limit is an idea that all other calculus concepts are based on. Limits involving infinity are closely related to asymptotes. While asymptotes for functions are sometimes easy to identify from a graph, the actual definitions of asymptotes are given in terms of limits. There are many different types of asymptotes and the two simplest ones are:

Asymptote	Equation	Definition
Horizontal	$y = L$	$\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$
Vertical	$x = a$	$\lim_{x \rightarrow a^+} f(x) = \infty$ or $(-\infty)$ or $\lim_{x \rightarrow a^-} f(x) = \infty$ or $(-\infty)$

This lab is designed to provide experience finding those two types of asymptotes and limits involved. We will also learn to use Maple to evaluate limits in general. Please use it as a learning tool, for example, to get a hint or check your homework answers, as Maple is likely not allowed in your calculus tests.

Maple Essentials

Important Maple commands introduced in this lab:

Command	Description	Example
<code>limit(f(x), x=a);</code>	evaluate $\lim_{x \rightarrow a} f(x)$	<code>limit(f(x), x=2); limit(x^2, x=infinity);</code>
<code>limit(f(x), x=a, right);</code>	evaluate $\lim_{x \rightarrow a^+} f(x)$	<code>limit((f(x)+1)/(x^2-1), x=1, right);</code>
<code>limit(f(x), x=a, left);</code>	evaluate $\lim_{x \rightarrow a^-} f(x)$	<code>limit(sqrt(x^2+1)/(x+1), x=-1, left);</code>
<code>solve</code>	solve equations	<code>solve(x^2=8, x); solve(denom(f(x))=0, x);</code>

Related course material/Preparation

§2.1, §2.2, and §2.3 (Pages 101-134) of the textbook (Anton, 8th edition).

Activities

A) Identify all horizontal and vertical asymptotes for functions on the back of this page.

General Directions

1. Define your function, say $f(x)$, using `f:=x->` and verify it is entered correctly (for example, you have to enter `3*x` for $3x$).
2. Look at your function $f(x)$ and determine which values make the denominator zero. These values will be the a 's at which we need to check for possible vertical asymptotes. If you need help, you can use the command `solve` as follows:
`> solve(denom(f(x))=0, x);`
 If you don't like to type the command, your TA will show you how to do it using the right-clicking.
3. You then need to find side-limits at $x = a$. Enter the following:
`> limit(f(x), x=a, left);`
`> limit(f(x), x=a, right);`
 If either of these returns the value ∞ or $-\infty$ then $x = a$ is the equation of a vertical asymptote of $f(x)$. Notice that $\lim_{x \rightarrow a} f(x)$ may or may not exist as two side-limits may or may not equal.

4. To find horizontal asymptotes of $f(x)$, you need to find limits as x goes to $+\infty$ or $-\infty$. Enter the following:
- ```
> limit(f(x), x=infinity);
> limit(f(x), x=-infinity);
```
- If either of these returns a value  $L \neq \pm\infty$  then  $y = L$  is the equation of a horizontal asymptote of  $f(x)$ . Notice that you may have two different horizontal asymptotes of  $f(x)$ .
5. Finally, plot the function and verify your answers geometrically. Since functions blow up near their vertical asymptotes, you need to specify appropriate ranges for both  $x$  and  $y$  in order to get nice looking graphs that includes all asymptotes.

### Functions

1.  $f1(x) = \frac{3x^2 + 2x - 1}{x + 2}$
2.  $f2(x) = \frac{2x + 5}{(x + 1)^2}$
3.  $f3(x) = \frac{(x^2 - 1)\sqrt{4x^2 + 1}}{x^3 - 2x^2 - x + 2}$
4.  $f4(x) = \frac{\sin x}{x}$
5.  $f5(x) = \frac{x^3 + 3x^2 - 12x + 4}{x^2}$
6.  $f6(x) = \frac{x^3 + 3x^2 - 12x + 4}{x^3 - x^2 - 4x + 4}$
7.  $f7(x) = \frac{\sqrt{x^2 + 4} - 2}{x}$
8.  $f8(x) = \frac{\sqrt{x^2 + 1} + 2x}{x}$

B) Use Maple to find  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  for the following  $f(x)$  (you will soon learn that this limit defines the derivative of  $f(x)$ ):

1.  $f(x) = x^2$
2.  $f(x) = 1/x$
3.  $f(x) = \sin x$

C) If you have time left, use the Maple to check answers for some home work problems on limits.

### Assignment

- 1) Identify all horizontal and vertical asymptotes for  $f5(x)$  to  $f8(x)$  on this page.
- 2) Answer the following questions (you may want to verify your answers with functions on this page):
  - If  $f(x)$  has a hole or a vertical asymptote at  $x = a$ , does that mean  $\lim_{x \rightarrow a} f(x) = \infty$  or  $-\infty$ ?
  - Does every hole in the domain of a function lead to a vertical asymptote?
  - Can the graph of a function cross the graph of its horizontal asymptotes? Its vertical asymptotes?
  - How many horizontal asymptotes can a graph have?