

# Newton's Method

Douglas Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

## Overview

The analysis of a function via calculus involves solving a variety of equations:  $f(x) = 0$  for zeros,  $f'(x) = 0$  for critical points,  $f''(x) = 0$  for possible inflection points. These equations are generally easy to formulate but difficult to solve. In fact, in many cases it is impossible to find exact solutions. Newton's Method is one algorithm for finding an approximate solution. In fact, it is one part of the algorithm used in Maple's `fsolve` command introduced in the lab for Week 8 (*Implicit Differentiation*).

## Maple Essentials

- The *Newton's Method* tutor in Maple 9.5 can be found under the **Tools** menu:

**Tools** → **Tutors** → **Calculus – Single Variable** → **Newton's Method ...**

## Related course material

§5.6 Newton's Method. Please read it before writing your report of project 2.

## Introduction to Newton's Method

The basic ideas behind finding an approximate solution to the equation  $F(x) = 0$  by Newton's Method are as follows:

- Pick a starting number,  $x_0$ , near the solution to  $F(x) = 0$ .  
*One possibility is to use the graph to locate a number close to the desired solution to  $F(x) = 0$ .*
- Compute  $x_1$ , a number would be closer to the solution of  $F(x) = 0$ , as follows:
  - Construct the tangent line  $y = F(x_0) + F'(x_0)(x - x_0)$  to the graph of  $y = F(x)$  at  $x = x_0$ .
  - The  $x$ -intercept of the above tangent line is hence given by  $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$ .  
*Observe that the  $x$ -intercept of the tangent line is generally closer to the solution than is  $x_0$ .*
- For  $n = 1, 2, 3, \dots$ , repeat this process using  $x_n$  as a new starting point and yielding  $x_{n+1}$  as an improved estimate to the solution of  $F(x) = 0$ ; the general formula for each step in Newton's Method is  $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$ .
- End the iteration, typically after a fixed number of iterations or when the estimates appear to stabilize.

## Activities

- Use Newton's method to estimate all solutions of  $x^3 - 5x = -1$ .  
*Remember to enter the initial guess as a floating-point number.*
- Use Newton's Method to estimate all critical numbers of  $f(x) = x^{\sin x}$  on  $[0, 10]$ .
- Explain the behavior of Newton's Method for  $F(x) = \frac{x}{x^2+1}$  with  $x_0 = 2$ ,  $x_0 = 1$ , and  $x_0 = \frac{1}{2}$ .

## General directions for problem 1, using the Newton's Method Tutor:

We will rewrite the problem as a root-finding problem: solve  $x^3 - 5x + 1 = 0$ .

- First estimates roots from an appropriate plot. (How many real-valued solutions are there? Use the plot to estimate each of these solutions.)
- Launch the *Newton's Method* tutor. Enter the function as `x^3-5*x+1`, the initial guess as `1`, and the number of iterations as `1`; press the **Display** button. This computes and shows the first iteration of Newton's Method. Change the number of iterations to `2` and press the **Display** button; this displays the first two iterations of Newton's Method. Increase the number of iterations until at least 5 digits to the right of the decimal point appear to stabilize.

3. Change the initial guess to 2 and apply Newton's Method enough times to estimate another root with an accuracy of at least 4 digits to the right of the decimal point.
4. Repeat the previous step with the initial guess set to -1. Note that Newton's Method does not always converge to the closest root.
5. Find an initial guess so that Newton's Method converges to the remaining root of this function.
6. Compare your estimates to those obtained with `fsolve(x^3-5*x=-1)`;
7. The maple command for Newton's Method is `NewtonMethod` (with `Student[Calculus1]` package). It can do some nice things with different options. For example, with `output=plot` it will generate nice supporting graphs. Try the following example:

```
with(Student[Calculus1]):
```

```
NewtonMethod(x^3-x,x=2,view=[0..3,DEFAULT],output = plot);
```

For more details, use maple's help menu.

### Implementing Newton's Method in Maple

- Basic Newton Iteration

The following command defines a new Maple command, `Newton`, that computes the next iterate in Newton's Method for solving  $F(x) = 0$  with current guess  $x = x_0$ :

```
> Newton := (F,x0) -> eval(x-F/diff(F,x),x=x0);
```

- Step-by-Step Iteration

```
> F :=x-> x^3-5*x+1; # define function to be solved for zeros
> Newton( F(x), 0.0 ); # computes x1 from x0 = 0(use 0.0 not 0)
> Newton( F(x), % ); # re-execute as often as needed
```

- Fixed Number of Iterations

```
> x[0] := 2.0; # initial guess is x0 = 2
> for n from 0 to 5 do # beginning of loop (6 iterations)
> x[n+1] := Newton( F(x), x[n] ); # computes xn+1 from xn
> end do; # end of loop
```

- Iterate Until Tolerance Met

```
> Digits := 20; # number of digits used in calculations
> tol := 5*10^(-9); # tolerance to get 8 decimal digits correct
> x[0] := -2.0; # initial guess is x0 = -2
> err[0] := 1; # initial error (anything > tol)
> for n from 0 to 49 while (err[n]>tol) do # loop until tolerance is met (50 iter max)
> x[n+1] := Newton( F(x), x[n] ); # computes xn+1 from xn
> err[n+1] := abs( x[n+1]-x[n] ); # update error
> end do; # end of loop
```

### Assignment/Project

Write a comprehensive report on Newton's Method. This will be Project 2 and is due at the beginning of next week's lab period. Your report should contain (but not restricted) the following information:

- An introduction of the method and explanations of mathematical ideas behind it. Adding some notes on motives and histories would be nice.
- Analysis of the method on, for example, its efficiency and difficulties.
- How to implement the method in Maple.
- Applications, examples, and supporting graphs.
- A summary.

Be creative in your presentation. If you use a graph or a table, remember that the main text needs to explain what information is contained. All floating point numbers should be reported with an accuracy of at least four digits to the right of the decimal point. Consult with your TA if you have any questions about the project.