Mathematical Models-Design a Roller Coaster

Douglas Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

Overview

There are three main objectives in this lab

- understand the mathematical reasoning associated with a real-world example,
- learn to define a piecewise-defined function in Maple, and
- learn to set up and to solve a system of equations in Maple.

Maple Essentials

- Important Maple command introduced in this lab are:

Command	Description	Example
`piecewise`	piecewise-defined function	`piecewise(x<=0,2*x,x>0 and x<2,x^2);`
`assign`	assignment	`assign({x=3,y=1}); assign(solutions);`

Design A Roller Coaster-An Example

Suppose we are asked to design a simple ascent and drop roller coaster. By studying photographs of our favorite coasters, we decide it should first ascend along a straight line $y = f1(x)$ of the slope 1.5 with a horizontal displacement of 20ft. It should then continue ascent and drop along a parabola $y = f2(x) = ax^2 + bx + c$ for another 100ft horizontally. Finally, we decide it should start a soft landing at 30ft above the ground along a cubic $y = f3(x) = dx^3 + ex^2 + fx + g$ for the last 80ft.

Here are our tasks:

1. Find equations in $\{a, b, c, d, e, f, g\}$ that will ensure the track is smooth at transition points.
2. Solve the equations in part (1) to find our functions.
3. Plot the graph to see the design.
4. Find the maximum height of the roller coaster.

Related course material/Preparation

Since our roller coaster consists of three curves, it can be set up mathematically as a piecewise-defined function $F(x)$. Let us use the origin as the starting point. We then have

$$F(x) = \begin{cases} f1(x), & 0 \le x \le 20 \\ f2(x), & 20 < x < 120 \\ f3(x), & 120 \le x \le 200 \end{cases}$$

To ensure our track is smooth at transition points, we must find constants $\{a, b, c, d, e, f, g\}$ such that $F(x)$ is continuous and differentiable for all x. In particular, $\lim_{x\to 20^+} F(x) = f2(20)$ must be equal to $\lim_{x\to 20^-} F(x) = f1(20)$ and $f2'(20) = 1.5$ (since $f1'(20) = 1.5$). Similarly, at $x = 120$, we need $f3(120) = f2(120)$ and $f3'(120) = f2'(120)$. Moreover, to start the soft landing at 30ft above the ground for the last 80ft, we must set $f3(120) = 30$. Finally, in order to have a soft landing, the tack should be tangent to the ground at the end, that is, $f3(200) = 0$ and $f3'(200) = 0$. We hence have a system of 7 equations. Since it would be very hard to solve it by hand, we will let Maple do the job. Notice that we should get a unique set of solutions, as the number of equations (7) is equal to the number of constants to be solved.

Solving the Problem

1. Start a Maple session and type `restart;` (this clears the internal memory so that Maple acts (almost) as if just started and is very helpful in case that you made a mistake and want to start over.)
2. It is easy to see that $f1(x) = 1.5x$ so **define** it with:
   `f1:=x->1.5*x;`
3. **Define** $f2(x)$ and $f3(x)$ with:
   `f2:=x->a*x^2+b*x+c;`
   `f3:=x->d*x^3+e*x^2+f*x+g;`
4. Now, **define** the piecewise function $F(x)$ with:
   `F:=x->piecewise(x<=20, f1(x),x>20 and x<120,f2(x),x>=120,f3(x));`
5. Verify your $F(x)$ with:
   `F(x);`
6. Find derivatives $f2(x)$ and $f3(x)$ and **assign** them to $df2$ and $df3$ with:
   `df2:=diff(f2(x),x);`
   `df3:=diff(f3(x),x);`
7. Assign seven equations as stated in Preparation. The first two in Maple are given below and you need to complete the step with the other five.
   `eq1:=f2(20)=f1(20);`
   `eq2:=eval(df2,x=20)=1.5;`
8. Solve the system of seven equations obtained in step 7 using the `solve` command and assign the solutions to values with:
   `values:=solve({eq1,eq2,eq3,eq4,eq5,eq6,eq7},{a,b,c,d,e,f,g});`
9. Plug those solved values into functions once for all with:
   `assign(values);`
10. Find derivative of the solved $F(x)$ and **assign** it to dF with:
    `dF:=diff(F(x),x);`
11. Take a look of your roller coaster with (notice that we want the same scale for both x and y):
    `plot(F(x),x=0..200,y=-50..150);`
12. To find the maximum height of the roller coaster, notice that it can only occur at points where the graph has a horizontal tangent line (why?). Find those points with:
    `solve(dF=0,x);`
13. To find the maximum height, check values of $F(x)$ at those points.
14. Remember to save before log out.

Assignment

Your assignment is to repeat this lab for the following problem and to **prepare a neat and complete project report**. All projects are due at the beginning of next week's lab.

Design A Bigger Roller Coaster-Project 1

You are now asked to design a bigger roller coaster which extends 400ft horizontally. It should ascend along a straight line $y = f1(x)$ of the slope 2 for the first 20ft. It then continues along three cubics $f2(x) = ax^3 + bx^2 + cx + d$, $f3(x) = ex^3 + fx^2 + gx + h$, and $f4(x) = ix^3 + jx^2 + kx + l$ for 100ft each. In addition, it needs to be 100ft above the ground at 80ft mark. It should also reach a bottom of 10ft above the ground and a top of 65ft above the ground at 180ft mark and 260ft mark, respectively. Finally, it should start a soft landing at 20ft above the ground along a cubic $f5(x) = mx^3+nx^2+ox+p$ for the last 80ft. (Hint: You should have 16 equations now instead of 7.)