

The ϵ - δ Definition of Limits

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Overview

The rigorous ϵ - δ definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts. We will also learn to use Maple to solve inequalities and equations.

Maple Essentials

- Important Maple commands introduced in this lab are:

Command	Description	Example
<code>solve(f(x), x);</code>	solve for x from f(x)	<code>solve(abs(x-1)<1); solve(x^2=a, x);</code>
<code>f(x) assuming p;</code>	assuming p in f(x)	<code>solve(abs(x)=a, x) assuming a>0;</code>

- The *EpsilonDelta* maplet is started from the course website:
– www.math.sc.edu/calclab/141L-F05/labs/ → [EpsilonDelta\(USC\)](#)

Related course material

§2.4 (Pages 134-143) of the textbook. Let us first recall the definition of limit given there: Let $f(x)$ be defined for all x in some open interval containing the number a , with the possible exception that $f(x)$ need not be defined at a . We will write

$$\lim_{x \rightarrow a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - a| < \delta.$$

In general, ϵ and δ meant to be very small numbers. Therefore, intuitively, it states that $f(x)$ will be very very close to L that is, $|f(x) - L| < \epsilon$, when x is very very close to a ($|x - a| < \delta$). The task is to show that, for any given ϵ (no matter how close $f(x)$ is to L), you can always find a δ -needed closeness of x to a -to make it work.

Activities

From our discussion, our job is to find a δ for a given ϵ , that is, solve for a range $a - \delta < x < a + \delta$ of x from the given inequality $|f(x) - L| < \epsilon$. Ideally, we would like to find a formula of δ in terms of ϵ (see examples 1, 2, and 3 of §2.4) that will work for any given ϵ . However, such formulas are in general very hard to find. Moreover, the value of δ is not unique. Maple's `solve` command can help us to find the largest δ that works for a given ϵ and the interactive *EpsilonDelta* maplet provides a tool to visualize relations between δ and ϵ ,

A) For each of the limits on the back, your task will be to discover the δ for each given ϵ .

General Directions

1. Look at the limit and identify $f(x)$, L , a , and ϵ .
2. Launch the *EpsilonDelta* maplet.
3. Enter the function $f(x)$, a , and L . Click plot setup to enter an appropriate viewing window. Enter ϵ .
4. You should see the graph of $f(x)$ in blue with a pink strip that goes from $a - \delta$ to $a + \delta$ along the x -axis and a brown strip that goes from $f(-\delta)$ to $f(x + \delta)$ along the y -axis. You will notice two red horizontal lines, one at $L - \epsilon$ and the other at $L + \epsilon$.
5. Your task is to increase the value of δ as far as possible without extending the brown stripe beyond the red lines. You should zoom several times to insure that you have not crossed either horizontal line.
6. When you think you are done, write down your last value of δ that did not cross the line.
7. Now we will find the value of δ more precisely using the **solve** command as follows

$$\text{solve}(\text{abs}(f(x)-L) < \epsilon, x);$$

where $f(x)$, L and ϵ are all replaced with your values. Maple will return an open interval.

8. Find the distances from a to the left bound and from a to the right bound of the interval. The *smallest* of these two values is the *largest* δ that works for this ϵ .
9. Compare your values from the *EpsilonDelta* maplet and from using the **solve** command. They should be very close.

Functions

1. $\lim_{x \rightarrow 9} \sqrt{x} = 3$, $\epsilon = 0.15$, $\epsilon = 0.05$
2. $\lim_{x \rightarrow 5} 1/x = 0.2$, $\epsilon = 0.1$, $\epsilon = 0.05$
3. $\lim_{x \rightarrow 3} (5x - 2) = 13$, $\epsilon = 0.05$, $\epsilon = 0.01$
4. $\lim_{x \rightarrow 2} (x^2 + 3x - 1) = 9$, $\epsilon = 0.8$, $\epsilon = 0.6$

B) For some simple functions like linear functions, **solve** can be used to find general formulas of δ in term of ϵ . Try the following and compare it to problem 3:

$$\text{solve}(\text{abs}(5*x-2-13)<\epsilon,x) \text{ assuming } \epsilon>0;$$

C) If you have time left, find out more about command **solve** using Maple's help menu and try some examples given there.

Assignment

Exercises 5,6,7,and 8 in §2.4 on page 141.