2.2 The stable category of modules

Def 2.2.1 A ring R is a (left) **Frobenius ring** if

R- projective module $\Leftrightarrow$ R- injective module.

Ex: ① $k[G]$, where k = field, G = finite group. (Representation Theorey)

② a finite graded connected Hopf algebra over a field.

Def 2.2.2 Suppose R is a ring. Given $f, g : {}_RM \to {}_RN$.

f is **stably equivalent to** g, written $f \sim g$,

if $f-g$ factors through a projective module.

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr, dashed, "\alpha"'] & & N \\ & P \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

Rmk: Interested in Frobenius rings.

Lemma 2.2.3 Stable equivalence is an equivalence relation compatible with composition. i.e., $f \sim g \Rightarrow hf \sim hg$, $fh \sim gh$.

pf:

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr] & & N \\ & P \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} M \arrow[rr, "g-h"] \arrow[dr] & & N \\ & Q \arrow[ur] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} M \arrow[rr, "f-h"] \arrow[dr] & & N \\ & P \oplus Q \arrow[ur] & \end{tikzcd}$$

Def : 2.2.4 Let R be a ring. The **stable category** of R-modules is the category whose objects are left R-modules whose morphisms are stable equivalence classes of R-mod morphisms.

f is a **stable equivalence** if it's an isomorphism in the stable category.

Note : Stable equivalences are closed under retracts and satisfy 2-3.

retract :

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, shift left, "g"] & A \arrow[d] \\
B \arrow[r] \arrow[d] & D \arrow[r, "P"'] \arrow[u, dashed, shift left, "g^{-1}"] \arrow[d] & B \arrow[d] \\
A \arrow[r] & C \arrow[r] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend right] & C \arrow[r] & P \arrow[r] \arrow[rr, bend right] & C \arrow[r] & A
\end{tikzcd}$$

2-3 : similar

Goal : When R is a Frobenius ring, stable category of R-modules is the homotopy category of a model structure on R-mod

Hence, we need a set I of generating cofibrations, a set J of generating trivial fibrations.

Def 2.2.5 Suppose R is a Frobenius ring. Let I denote the set of inclusions $a \to R$, where a is a left ideal; Let J denote the set (singleton) of inclusion $0 \to R$

Define a map of R-modules to be a **fibration** if it has the right lifting property with respect to J.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ R \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

f is a **cofibration** if. $f \in I\text{-cof}$

Recall

$f \in I\text{-inj}$:

$$\begin{tikzcd} a \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ R \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

for all $a \to R \in I$

$g \in I\text{-cof}$:

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "g"'] & M \arrow[d, "f"] \\ N' \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

for all $f \in I\text{-inj}$

Goal: cofibrations, fib, stable equivalences

define a model structure on R-mod.

Using Thm 2.1.19

Lemma 2.2.6 A map p in R-mod is a fibration $\iff$ p is surjective.

pf:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ R \arrow[r] \arrow[ur, dashed] & N \end{tikzcd} \quad \iff \quad \begin{tikzcd} & R \arrow[d] \arrow[dl, dashed] \\ M \arrow[r, "p"] & N \end{tikzcd}$$

R = projective

$\square$

Lemma 2.2.7 Suppose R is a Frobenius ring. A map f in R-mod is a trivial fibration $\iff$ f is surjective with projective kernel.

pf: $\Rightarrow$: If $f: M \to N$ is a stable equivalence,

then $\exists\, q: N \to M$ s.t.

$$\begin{tikzcd} M \arrow[d, "p"] \\ N \arrow[d, "q"] \\ M \end{tikzcd} \qquad \begin{tikzcd} N \arrow[d, "q"] \arrow[dr, "i"] & \\ M \arrow[d, "p"] & P \arrow[dl, "h"] \\ N & \end{tikzcd}$$

and $pq - i_N = hi$

consider

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] \arrow[d, "a"] & M \arrow[r, "f"] \arrow[d, "b"] & N \arrow[r] \arrow[d, equal, "c"] & 0 \\ 0 \arrow[r] & Q \arrow[r] & M \oplus P \arrow[r, "f \oplus h"'] & N \arrow[r] & 0 \end{tikzcd}$$

① claim Q is projective

② claim $\ker f$ is a retract of Q, hence projective.

① $M \oplus P \xrightarrow{f \oplus h} N$, $N \xrightarrow{(q,-i)} M \oplus P$

claim $N \xrightarrow{(q,-i)} M \oplus P \xrightarrow{f \oplus h} N$ is id_N

$n \mapsto (q(n), -i(n)) \longmapsto fq(n) - hi(n) = n$

so the lower exact sequence splits.

$$\begin{tikzcd} M \arrow[r, "stable"] \arrow[d, "stable"] & N \arrow[d, "stable"] \\ M \oplus P \arrow[r, "f \oplus h"'] & N \end{tikzcd}$$

so $f \oplus h$ is stable.

$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & M \oplus P \arrow[r] & N \arrow[r] & M \oplus P \\ & \widetilde{P} \arrow[urr] & & \end{tikzcd}$$

hence the inclusion $Q \longrightarrow M \oplus P$ is stable zero

Thus
$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & M \oplus P \arrow[r] & Q \\ & \widehat{P} \arrow[ur] & \end{tikzcd}$$

Therefore Q is projective.

② By snake lemma, $0 \to \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c \to 0$

~~i.e., $0 \to \ker f \to$~~ $0 \to Q/\ker f \to P \to 0$

so $Q/\ker f$ is projective.

Then $Q/\ker f$ so $Q = \ker f \oplus Q/\ker f$

($\Leftarrow$)

$$0 \to \underset{\text{injective}}{I} \rightleftarrows M \xrightarrow{f} N \xrightarrow{\oplus} 0$$

so splits ~~so $\exists g: N \to M$ s.t $fg = 0$~~

$$M = I \oplus N = P \oplus N \quad \text{so} \ldots$$

Lemma 2.2.8 An R-module Q is ~~projective~~ injective $\Leftarrow$ for all left ideals a.

$$\begin{tikzcd} & Q \\ 0 \arrow[r] & a \arrow[r] \arrow[u] & R \arrow[ul, dashed] \end{tikzcd}$$

Prop 2.2.9 If R is an arbitrary ring, P is I-inj

$\Leftarrow$ P is a surjection with injective kernel

In particular, if R is a Frobenius ring, the trivial fibrations form the class I-inj

~~Rmk~~

pf: ($\Leftarrow$) Suppose $p: M \to N$ is a surjection with injective kernel.

and we have
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & M \arrow[d, "p"] \\ R \arrow[r, "g"'] & N \end{tikzcd}$$

$\ker P$ is injective, so splits, hence

$$\begin{tikzcd} & \ker P \arrow[d] \\ a \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ R \arrow[r] & N \arrow[u, bend left, "q"] \arrow[d] \\ & 0 \end{tikzcd}$$

$\exists\ q: N \to M$ s.t $pq = i_N$

consider the map $qgi - f : a \to M$

$$\begin{tikzcd} & \ker p \arrow[d, "j"] \\ a \arrow[r, "f"] \arrow[d, "i"'] & M \arrow[d, shift left, "p"] \\ R \arrow[r, "g"'] & N \arrow[u, shift left, "q"] \end{tikzcd}$$

$$p(qgi - f) = ig - pf = 0$$

hence $\mathrm{Im}(qgi - f) \subseteq \ker p$

let $jr := qgi - f : a \xrightarrow{r} \ker p \xrightarrow{j} M$

since $\ker p$ is injective:

$$\begin{tikzcd} & & \ker p \\ 0 \arrow[r] & a \arrow[ur, "r"] \arrow[r, "i"'] & R \arrow[u, dashed, "s"'] \end{tikzcd}$$

$\exists\ s : R \to \ker p$. and $r = si$

we have : $R \xrightarrow{s} \ker p \circ \xrightarrow{j} M$

~~and $pjsi = pjr = p(qgi - f) =$~~

~~(1) $jsi = jr = qgi - f$~~

we have ~~Define~~ the lift $qg - js : R \to M$

so $p \in I\text{-inj}$

($\Rightarrow$) suppose $p \in I\text{-inj}$.

Since $J \subseteq I$, we have $p \in J\text{-inj}$. so p is surjective

We need to show $\ker p$ is an injective R-module.

suppose a is a left ideal of R. $f: a \to \ker p$ a hom..

The we have

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "i"'] & \ker p \arrow[r, "j"] & M \arrow[d, "p"] \\
R \arrow[ru, dashed, "h"] \arrow[rru, dashed, "h"'] \arrow[rr, "0"'] & & N
\end{tikzcd}$$

p is in $I\text{-inj}$ $\Rightarrow$ $\exists$ $h: R \to M$ with $ph = 0$

so $\operatorname{Im} h \subseteq \ker p$

hence $h: R \to \ker p$

Rmk: If R is a Frobenius ring, we have two ways to define trivial fibrations
$\begin{cases} ① \ I\text{-inj} \\ ② \ \text{lift } \begin{smallmatrix} 0 \\ \downarrow \\ R \end{smallmatrix} \text{ and stable.} \end{cases}$

Lemma 2.2.10 Over an arbitrary ring R, I-cof is the class of injections.

pf: $f \in I\text{-cof} \iff f$ has the left property with respect to all surjections with injective kernel.

Similar to 2.2.9

Lemma 2.2.11 Over an arbitrary ring R, a map f is in J-cof $\iff f$ is an injection with projective cokernel.

In particular, the elements of J-cof are stable equivalences.

pf: $(\Leftarrow)$

$$\begin{tikzcd} 0 \arrow[d] & \\ A \arrow[r, "f"] \arrow[d, "i"'] & M \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[u, dashed, bend right, "r"'] \arrow[d, "j"] & N \\ C \arrow[d] \arrow[uur, dashed, "t"] \arrow[ur, dashed, "s"'] & \\ 0 & \end{tikzcd}$$

p = surjection

Suppose we have the commutative diagram, $C = \operatorname{coker} i$ is projective

so we can reverse $i: A \to B$ to get $r: B \to A$

Consider $pfr - g : \overset{A \oplus C}{\overset{\shortparallel}{B}} \to N$, we have

$$(pfr-g)i = pfri - gi = pf - gi = 0$$

i.e $pfr - g|_A = 0$, hence we have a map

$s: C \to N$,

C is projective, so $\exists\ t: C \to M$ s.t $pt = s$ and $\boxed{fr - tj : B \to M}$

($\Rightarrow$) Suppose $i: A \to B$ is in J-cof, hence in I-cof

$\left(J \subseteq I \Rightarrow J\text{-inj} \supseteq I\text{-inj} \Rightarrow J\text{-cof} \subseteq I\text{-cof} \right)$

so i is an injection.

Now consider the diagram below. derived from $\begin{matrix} & & C & \\ & & \downarrow f & \\ M & \to & N & \to 0 \end{matrix}$

$$\begin{tikzcd}
0 \arrow[d] & \\
A \arrow[r, "0"] \arrow[d, "i"'] & M \arrow[d, "p"] \\
B \arrow[r, "fj"] \arrow[d, "j"'] \arrow[ur, dashed, "h"] & N \\
C \arrow[ur, "f"'] \arrow[d] & \\
0 &
\end{tikzcd}$$

p : surjection

and we get a lift $h: B \to M$

Notice $hi = 0$ so ~~$\exists g$~~ $A \subseteq \ker h$.

so $\exists\, g: B/A \to M$ i.e., $g: C \to M$

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Thm 2.2.12 Suppose R is a Frobenius ring. Then there is a cofibrantly generated model category structure on R-mod where cofibrations = injections. fib = surjections, weak equivalence = stable equivalence.

Moreover, ~~If~~ R is Noetherian, so ~~then~~ the module structure above is finitely generate

Pf: By 2.1.6, every R-module is small.

Apply 2.1.19.

Frobenius $\Rightarrow$ Noetherian