

## 2.2 The stable category of modules

Def 2.2.1 A ring  $R$  is a (left) **Frobenius ring** if

$R$ -projective module  $\Leftrightarrow R$ -injective module.

Ex: ①  $k[G]$ , where  $k = \text{field}$ ,  $G = \bullet$  finite group. (Representation Theory)

② a finite graded connected Hopf algebra over a field.

Def 2.2.2 Suppose  $R$  is a ring. Given  $f, g: {}_R M \rightarrow {}_R N$ .

$f$  is **stably equivalent to**  $g$ , written  $f \sim g$ ,

if  $f - g$  factors through a projective module.

$$\begin{array}{ccc} M & \xrightarrow{f-g} & N \\ \vdots & & \nearrow \\ \downarrow & \dashrightarrow & \downarrow \\ & P & B \end{array} \quad \textcircled{P}$$

Rmk: Interested in Frobenius rings.

Lemma 2.2.3 Stable equivalence is an equivalence relation compatible

with composition. i.e.,  $f \sim g \Rightarrow hf \sim hg$ ,  $fh \sim gh$ .

pf:

$$\begin{array}{ccc} M \xrightarrow{f-g} N & & M \xrightarrow{g-h} N \\ \downarrow \quad \uparrow & & \downarrow \quad \uparrow \\ P & & Q \end{array} \Rightarrow \begin{array}{ccc} M \xrightarrow{f-h} N & & \\ \downarrow \quad \uparrow & & \\ P \oplus Q & & \end{array}$$

Def: 2.2.4 Let  $R$  be a ring. The **stable category** of  $R$ -modules is the category whose objects are left  $R$ -modules whose morphisms are stable equivalence classes of  $R$ -mod morphisms.  $f$  is a **stable equivalence** if it's an isomorphism in the stable category.

Note: Stable equivalences are closed under retracts and satisfy 2-3.

retract:

$$\begin{array}{ccccccc}
 A & \longrightarrow & C & \longrightarrow & A & & \\
 \downarrow & & \downarrow & \searrow & \downarrow & & \\
 B & \longrightarrow & D & \xrightarrow{p} & B & & \\
 \downarrow & & \downarrow & \swarrow & \downarrow & & \\
 A & \longrightarrow & C & \longrightarrow & A & & 
 \end{array}$$

$$\begin{array}{ccccccc}
 A & \longrightarrow & C & \longrightarrow & P & \longrightarrow & C & \longrightarrow & A \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & & & & & & & 
 \end{array}$$

2-3: similar

Goal: When  $R$  is a Frobenius ring, stable category of  $R$ -modules.

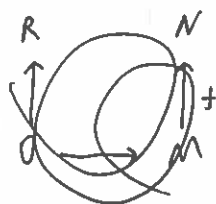
is the homotopy category of a model structure on  $R$ -mod

Hence, we need a set  $I$  of generating cofibrations,

a set  $J$  of generating trivial fibrations.

Def 2.2.5 Suppose  $R$  is a Frobenius ring. Let  $I$  denote the set of inclusions  $a \rightarrow R$ , where  $a$  is a left ideal; Let  $J$  denote the set (singleton) of inclusions  $0 \rightarrow R$

Define a map of  $R$ -modules to be a **fibration** if it has the right lifting property with respect to  $J$ .



$f$  is a **cofibration** if  $f \in I\text{-cof}$

Recall

$f \in I\text{-inj}$ :  $\begin{array}{ccc} a & \longrightarrow & M \\ \downarrow & \nearrow & \downarrow f \\ R & \longrightarrow & N \end{array}$  for all  $a \rightarrow R \in I$

$g \in I\text{-cof}$ :  $\begin{array}{ccc} M' & \longrightarrow & M \\ g \downarrow & \nearrow & \downarrow f \\ N & \longrightarrow & N \end{array}$  for all  $f \in I\text{-inj}$

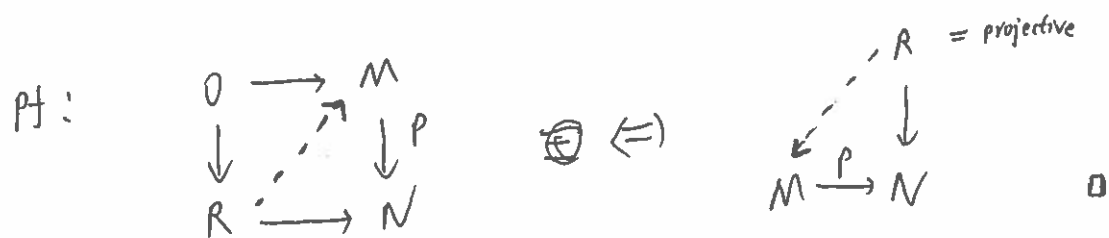
Goal: cofibrations, fib, stable equivalences

define a model structure on  $R\text{-mod}$ .

Using Thm 2.1.19

Lemma 2.2.6 A map  $P$  in  $R\text{-mod}$  is a fibration

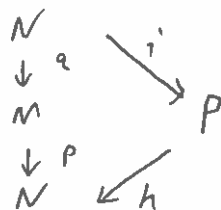
$\Leftrightarrow P$  is surjective.



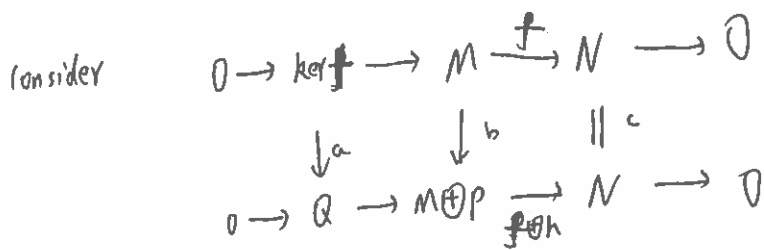
Lemma 2.2.7 Suppose  $R$  is a Frobenius ring. A map  $f$  in  $R\text{-mod}$  is a trivial fibration  $\Leftrightarrow f$  is surjective and with projective kernel.

Pf:  $\Rightarrow$  If  $f: M \rightarrow N$  is a stable equivalence,

then  $\exists q: N \rightarrow M$  s.t.



and  $Pq - i_N = hi$



① claim  $Q$  is projective

② claim  $\ker f$  is a retract of  $Q$ , hence projective.

$$\textcircled{1} \quad M \oplus P \xrightarrow{f \oplus h} N, \quad N \xrightarrow{(q, -i)} M \oplus P$$

claim  $N \xrightarrow{(q, -i)} M \oplus P \xrightarrow{f \oplus h} N$  is  $\text{id}_N$

$$n \mapsto (q(n), -i(n)) \mapsto f(q(n)) - h(i(n)) = n$$

so the lower exact sequence splits.

$$\begin{array}{ccc} M & \xrightarrow{\text{stable}} & N \\ \downarrow \text{stable} & & \downarrow \text{stable} \\ M \oplus P & \xrightarrow{f \oplus h} & N \end{array} \quad \text{so } f \oplus h \text{ is stable.}$$

hence the inclusion  $Q \rightarrow M \oplus P$  is stable zero

$$\begin{array}{ccccc} Q & \rightarrow & M \oplus P & \rightarrow & N & \rightarrow & M \oplus P \\ & & & & \searrow & & \nearrow \\ & & & & \tilde{P} & & \end{array}$$

Thus

$$\begin{array}{ccccc} Q & \rightarrow & M \oplus P & \rightarrow & Q \\ & & \searrow & & \nearrow \\ & & \tilde{P} & & \end{array}$$

Therefore  $Q$  is projective.

$\textcircled{2}$  By snake lemma,  $0 \rightarrow \ker a \rightarrow \ker b \rightarrow \ker c \rightarrow \text{coker } a \rightarrow \text{coker } b \rightarrow \text{coker } c \rightarrow 0$

$$\text{i.e. } 0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \rightarrow Q/\ker f \rightarrow P \rightarrow 0$$

so  $Q/\ker f$  is projective.

Then

$$\begin{array}{c} Q/\ker f \\ \swarrow \quad \downarrow \\ \ker f \end{array}$$

so  $Q = \ker f \oplus Q/\ker f$

( $\Leftarrow$ )

$$0 \rightarrow I \xrightarrow{\text{injective}} M \xrightarrow{f} N \rightarrow 0$$

so splits ~~so  $\exists g: N \rightarrow M$  s.t.  $fg = 0$~~

$$M = I \oplus N = P \oplus N \quad \text{so } \dots$$

Lemma 2.2.8 An  $R$ -module  $Q$  is ~~projective~~ <sup>injective</sup>  $\Leftrightarrow$  for all left ideals  $a$ ,

$$0 \rightarrow a \rightarrow R \rightarrow Q$$

Prop 2.2.9 If  $R$  is an arbitrary ring,  $P$  is  $I$ -inj  $\Leftrightarrow$

$\Leftrightarrow P$  is a surjection with injective kernel

In particular, if  $R$  is a Frobenius ring, the trivial fibrations form the class  $I$ -inj

~~Proof~~

pf: ( $\Leftarrow$ ) Suppose  $P: M \rightarrow N$  is a surjection with injective kernel.

$$\text{and we have } \begin{array}{ccc} a & \xrightarrow{f} & M \\ \downarrow i & & \downarrow P \\ R & \xrightarrow{g} & N \end{array}$$

$\ker P$  is injective, so splits, hence

$$\begin{array}{ccc} a & \longrightarrow & M \\ \downarrow & & \downarrow P \\ R & \longrightarrow & N \\ & & \downarrow \end{array} \quad \exists \varphi: N \rightarrow M \text{ s.t. } P\varphi = i_N$$

consider the map  $qgi-t: a \rightarrow M$

$$\begin{array}{ccc}
 & \text{ker } P & \\
 & \downarrow j & \\
 a & \xrightarrow{t} & M \\
 i \downarrow & & \uparrow q \\
 R & \xrightarrow{g} & N \quad \text{---} \quad \text{---} \quad M
 \end{array}$$

$$p(qgi-t) = ig - pt = 0$$

hence  $\text{Im}(qgi-t) \subseteq \text{ker } P$

let  $j\gamma := qgi-t: a \xrightarrow{\gamma} \text{ker } P \xrightarrow{j} M$

since  $\text{ker } P$  is injective:

$$\begin{array}{ccc}
 & & \text{ker } P \\
 & \nearrow \gamma & \uparrow s \\
 0 \rightarrow a & \xrightarrow{i} & R
 \end{array}$$

$\exists s: R \rightarrow \text{ker } P$  and  $\gamma = si$

we have  $\circ: R \xrightarrow{s} \text{ker } P \xrightarrow{j} M$

and  ~~$pjsi = pjr = p(qgi-t) =$~~

~~$\textcircled{1} \quad \text{---} \quad jsr = jr = qgi-t$~~

we have ~~define~~ the lift  $qg-js: R \rightarrow M$

so  $P \in I\text{-inj}$

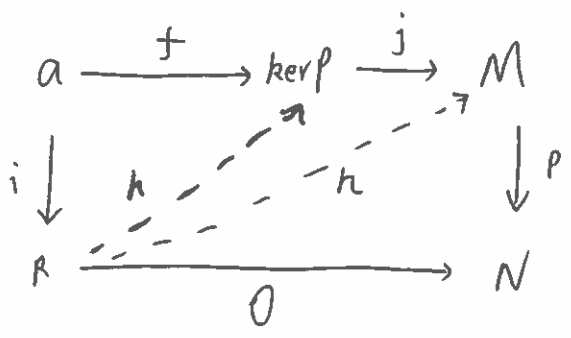
( $\Rightarrow$ ) suppose  $P \in I\text{-inj}$ .

Since  $J \subseteq I$ , we have  $P \in J\text{-inj}$ . so  $P$  is surjective

We need to show  $\ker P$  is an injective  $R$ -module.

suppose  $a$  is a left ideal of  $R$ ,  $f: a \rightarrow \ker P$  a hom.

Then we have



$P$  is in  $I\text{-inj} \Rightarrow \exists h: R \rightarrow M$  with  $Ph = 0$

so  $\text{Im } h \subseteq \ker P$

hence  $h: R \rightarrow \ker P$

Rmk: If  $R$  is a Frobenius ring, we have two ways to define trivial fibrations

- ①  $I\text{-inj}$
- ②  $\text{lift } \begin{array}{c} 0 \\ \downarrow \\ R \end{array}$  and stable.



Lemma 2.2.10 Over an arbitrary ring  $R$ ,  $I\text{-cof}$  is the class of injections.

Pf:  $f \in I\text{-cof} \Leftrightarrow f$  has the left property with respect to all surjections with injective kernel.

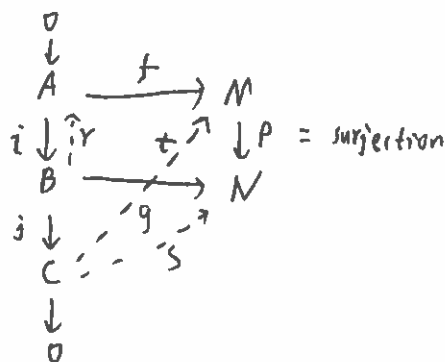
Similar to 2.2.9

Lemma 2.2.11 Over an arbitrary ring  $R$ , a map  $f$  is in  $J\text{-cof}$

$\Leftrightarrow f$  is an injection with projective cokernel.

In particular, the elements of  $J\text{-cof}$  are stable equivalences.

Pf: ( $\Leftarrow$ )



Suppose we have the commutative diagram,  $C = \text{coker } f$  is projective

so we can reverse  ~~$i: A \rightarrow B$~~  to get  $r: B \rightarrow A$

consider  $p+r-g: \overset{A \oplus C}{B} \rightarrow N$ , we have

$$(p+r-g)i = p+ri-gi = pf-gi = 0$$

i.e.  $p+r-g|_A = 0$ , hence we have a map

$$s: C \rightarrow N,$$

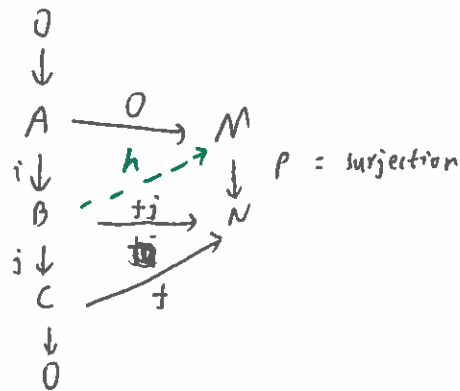
$C$  is projective, so  $\exists t: C \rightarrow M$  s.t.  $pt = s$  and  $f-r-tj: B \rightarrow M$

( $\Rightarrow$ ) Suppose  $i: A \rightarrow B$  is in  $J$ -cof, hence in  $I$ -cof

$$(J \subseteq I \Rightarrow J\text{-inj} \stackrel{2}{\subseteq} I\text{-inj} \stackrel{2}{\Rightarrow} J\text{-cof} \subseteq I\text{-cof})$$

so  $i$  is an injection.

Now consider the diagram below, derived from  $M \xrightarrow{c} N \rightarrow 0$



and we get a lift  $h: B \rightarrow M$

Notice  $hi = 0$  so  ~~$\exists g: B/A \rightarrow M$~~   $A \subseteq \ker h$ .

so  $\exists g: B/A \rightarrow M$  i.e.,  $g: C \rightarrow M$

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Thm 2.2.12 Suppose  $R$  is a Frobenius ring. Then there is

a cofibrantly generated model category structure on  $R\text{-mod}$

where cofibrations  $\cong$  injections, fib = surjections,

weak equivalence = stable equivalence.

Moreover,  ~~$\neq$~~   $R$  is Noetherian, so then the module structure above is finitely generate

Pf: By 2.1.6, every  $R$ -module is small.

Apply 2.1.19.

Frobenius  $\Rightarrow$  Noetherian