1. Model Categories

1.1 The Defination of a Model Category

Note: Given a category $\mathcal{C}$, we can form a new category $\mathrm{Map}\,\mathcal{C}$ whose objects are morphisms and whose morphisms are commutative squares.

Def: Suppose $\mathcal{C}$ is a category.

1. A map f is a retract of a map g (both in $\mathcal{C}$) if f is a retract of g as objects in $\mathrm{Map}\,\mathcal{C}$.
   i.e.,
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$
   where the horizontal composites are identities.

2. A functorial factorization is an ordered pair (∂, β) of functors $\mathrm{Map}\,\mathcal{C} \to$, such that $f = \beta(f) \circ \partial(f)$ for all $f \in \mathrm{Map}\,\mathcal{C}$. In particular, the domain of $\partial(f)$ is the domain of f, codomain $\partial(f)$ = domain $\beta(f)$, codomain $\beta(f)$ = cod

Rmk:
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\partial(f)"'] & & B \\ & C \arrow[ur, "\beta(f)"'] & \end{tikzcd}$$

$$\begin{tikzcd} & E \arrow[dr] & \\ A \arrow[ur] \arrow[rr, "f"] \arrow[d] & & B \arrow[d] \\ C \arrow[rr, "g"] \arrow[dr] & & D \\ & F \arrow[ur] & \end{tikzcd}$$
, we get a map $E \to F$ (functorial)

Def: Suppose $i: A \to B$ and $p: X \to Y$ are maps in a category, $\mathcal{C}$. Then i has the left lifting property with respect to p and p has the right lifting property with respect to i, if for every commutative diagram of the following form.
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

Map $\mathcal{C}$?
↑

A model structure on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties :

1. (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, so is the third.
2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.
3. (Lifting) Define a map to be a trivial cofibration if it's both a cofibration and a weak equivalence. Similarly, we can define trivial fibration. Then trivial cofibrations have the left lifting property with respect to fibrations and cofibrations ····· trivial fibrations.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{trivial cofibration}"'] & X \arrow[d, "\text{fibration}"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "\text{cofibration}"'] & X \arrow[d, "\text{trivial fibration}"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

4. (Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

A category is said to have all small limits (colimits) if, for every small category $\mathcal{I}$ and any functor $F : \mathcal{I} \to \mathcal{C}$, a limit (colimit) of F exists in $\mathcal{C}$.

A model category is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

Suppose $\mathcal{C}$ is a category with all small limits and colimits. Let

① weak equivalence ⟷ isomorphism ② fibration, cofibration ⟷ any map in $\mathcal{C}$.

③
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f = \alpha(f)"'] & & B \\ & B \arrow[ur, "\beta(f) = id_B"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "id_A = \alpha(f)"'] & & B \\ & A \arrow[ur, "\beta(f) = f"'] & \end{tikzcd}$$

Then this is a model category.

[redacted]

.: Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories. Then $\mathcal{C} \times \mathcal{D}$ becomes a model category in an obvious way:

① A map (f, g) is a fibration (cofibration, weak equivalence) if and only if f and g are both fibrations (cofibarations, weak equivalences)

②
$$\begin{tikzcd} A \times B \arrow[rr, "{(f,g)}"] \arrow[dr, "\alpha(f)\times\alpha'(g)"'] & & C \times D \\ & E \times F \arrow[ur, "\beta(f)\times\beta'(g)"'] & \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[rr, "{(f,g)}"] \arrow[dr, "\gamma(f)\times\gamma'(g)"'] & & C \times D \\ & E \times F \arrow[ur, "\delta(f)\times\delta'(g)"'] & \end{tikzcd}$$

: If $\mathcal{C}$ is a model category, then $\mathcal{C}^{op}$ is also a model category.

$$\mathcal{C} \longleftrightarrow \mathcal{C}^{op}$$

weak equivalence $\leftrightarrow$ weak equivalence

fibration $\leftrightarrow$ cofibration

cofibration $\leftrightarrow$ fibration

$(\alpha, \beta) \leftrightarrow$ ~~(δ, γ)~~ $(\beta, \alpha) =$ new $(\tilde{\gamma}, \tilde{\delta})$

$(\gamma, \delta) \leftrightarrow (\delta, \gamma) =$ new $(\tilde{\alpha}, \tilde{\beta})$

This is the dual category of $\mathcal{C}$, denoted by $D\mathcal{C}$, and $D^2\mathcal{C} = \mathcal{C}$.

☆ Hence every theorem about model categories has a dual theorem.

k: If $\mathcal{C}$ is a model category, then it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of empty diagram.

: Let $\mathcal{C}$ be a model category. An object is ~~a fibrant~~ fibrant if the map from it to the terminal object is a fibration.

An object is cofibrant if the map from initial object to it is a cofibration.

We can a model category pointed if the map from initial to terminal is an isomorphism.

|: Given a ~~category~~ model category $\mathcal{C}$, define the category $\mathcal{C}_*$ to be the category under the terminal object. i.e., induced by h^*

object: $\begin{tikzcd} * \arrow[r, "v"] \arrow[d, "id"] & A \arrow[d, "f"] \\ * \arrow[r] & B \end{tikzcd}$ (f: morphism) p: $\overbrace{\mathcal{C}_*}^{\text{indexed by } \mathcal{J}}$ has arbitrary limits and colimits : $\varprojlim (* \to A_i) = (* \to \varprojlim A_i)$

$\varinjlim (* \to A_i) = * \to \varinjlim A_i$

f: limit :

$$\begin{tikzcd} * \arrow[d] \arrow[dr] & \\ A_j \arrow[r] & A_k \end{tikzcd} \Rightarrow \exists ! \ * \to \varprojlim A_i \Rightarrow \begin{tikzcd} * \to W \arrow[dr, dashed, "\exists !"] \arrow[ddr, bend right] \arrow[ddrr, bend left] & & \\ & * \to \varprojlim A_i \arrow[d] \arrow[dr] & \\ & (* \to A_j) \arrow[r] & (* \to A_k) \end{tikzcd}$$

? colimit : Add an extra initial object $*$ to $\mathcal{J}$, we get $\mathscr{J}$.
So $\exists\, u : * \to \varinjlim i$ (in $\mathscr{J}$), hence $\exists\, v : * \to \varinjlim A_i$ (in $\mathcal{C}_*$)

And similarly to the limit, we can prove $(* \to \varinjlim A_i)$ is $\varinjlim (* \to A_i)$

p: $\mathcal{C}_*$ is a pointed category, and $*$ is its zero element.

ition: In $\mathcal{C}_*$, $v : * \to X$ is an object, and we consider it as an object X along with a basepoint v.

k: There is an obvious functor $\mathcal{C} \to \mathcal{C}_*$, $X \mapsto (* \to X \sqcup *)$

x: The coproduct of $* \to X$ and $* \to Y$ in $\mathcal{C}_*$ is

$$(* \to X \sqcup Y) \Big/ \left(\begin{matrix} * \to X \to X \sqcup Y \\ \| \\ * \to Y \to X \sqcup Y \end{matrix} \right)$$

[redacted]

Suppose $\mathcal{C}$ is a model category. Define a map f in $\mathcal{C}_*$ to be a cofibration (fibration, weak equivalence) if and only if $U(f)$ is a co (f, we) in $\mathcal{C}$. Then $\mathcal{C}_*$ is a model category.

Check all the ~~conditions~~ axioms in the defination.

: 1. We can replace $*$ by A, to get $h^A: X \to Mor(A, X)$. and a ~~category~~ category under A.
2. Dually, we can get $h_A: X \to Mor(X, A)$.

Let $\mathcal{C}$ be a model category with initial object X.

$$\begin{tikzcd} \text{Initial} \arrow[rr, "f"] \arrow[dr, "\partial(f)=\partial"'] & & X \\ & QX \arrow[ur, "\beta=\beta(f)"'] & \end{tikzcd}$$

(Dually,
$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "r(g)"'] & & * \\ & RX \arrow[ur, "s(g)"'] & \end{tikzcd}$$
where X is the termial object)

Indeed, ∂ is a functor, and $\partial(id_X)$ is a cofibration, hence $QX = \partial(X)$ is cofibrant. We refer to Q as the **cofibrant replacement functor** of $\mathcal{C}$.
(Why is it a functor!) ✓

Dually, we say R is the **fibrant replacement functor** of $\mathcal{C}$.

: (The Retract Argument) Suppose we have a factorization $f = p \circ i$ in a category $\mathcal{C}$. and suppose f has the left lifting property with respect to p. then f is a retract of i. Dually, if f has the right lifting property with respect to i, then f is a retract of p.

:
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$
and
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, "id"'] \arrow[ur, "h"] & B \end{tikzcd}$$
hence
$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "f"'] & A \arrow[r, "id"] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "h"'] & C \arrow[r, "p"'] & B \end{tikzcd}$$

Simaler to the duality.

Suppose $\mathcal{C}$ is a model category. Then a map is a cofibration (or trival co) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, fibration (trivial fibration) $\Longleftrightarrow$ it has the right lifting property with respect to all trivial cofibrations (fibrations).

$\Rightarrow$: ✓

$\Leftarrow$: Suppose f has left lifting property with respect to all trivial fibrations.

Consider
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"', "\text{cofibration}"'] & & B \\ & C \arrow[ur, "p"', "\text{trivial fibration}"'] & \end{tikzcd}$$

so f is a retract of i.

Hence f is a cofibration.

Other situations are similar.

Suppose $\mathcal{C}$ is a model category. Then cofibrations (trivial cofibrations) are closed under pushouts, i.e.,

for
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed] & \\ & & T \end{tikzcd}$$

if f is a cofibration, then g is a cofibration.

Dually. If g is a fibration (trivial fibration), then so is f (for pullback)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & X \arrow[d, "h"] \\ B \arrow[r] \arrow[urr, dashed] & D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

ma: (Ken Brown's Lemma) Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with a subcategory of weak equivalences that satisfyies the 2-out-of-3 axiom. Suppose $F: \mathcal{C} \to \mathcal{D}$ is a functor that takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between ~~cofibrant~~ cofibrant objects to weak equivalences.

Dually, if $\{\text{trivial fibrations between fibrants}\} \xrightarrow{F} \{\text{weak equivalences}\}$, then $\{\text{weak equivalences between fibrant objects}\} \xrightarrow{F} \{\text{weak equivalences}\}$.

Pf: ~~First, we claim~~ Suppose $f: A \to B$ is a ~~weakness~~ weak equivalence between cofibrant objects.

First, we claim $A \to A \amalg B$ and $B \to A \amalg B$ are cofibrations

Since
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \amalg B \end{tikzcd}$$
is a pushout, and $0 \to B$ is a cofibrantion,

we have $A \to A \amalg B$ is a cofibration. Similarly, $B \to A \amalg B$ is a cofibration.

Next, consider
$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, "f"] & & \\ & A \amalg B \arrow[r, "\pi_B"] \arrow[d, "q"', "\text{cofibration}"] & B \\ B \arrow[ur, "i_B"] & C \arrow[ur, "p"] & \end{tikzcd}$$
P = trivial fibration

Claim 1. $q \circ i_A$ and $q \circ i_B$ are weak equivalence

$p \circ (q \circ i_A) = f$, p, f are weak equivalences $\Rightarrow$ $q \circ i_A$ is weak equivalence,

Similarly, $q \circ i_B$ ✓

Claim 2. i_A, q are cofibrations, so is $q \circ i_A$. Similarly, so is $q \circ i_B$
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[dd, "\text{any trivial fibration}"] \\ A \amalg B \arrow[ur, dashed] \arrow[d] \arrow[dr, bend right=60] & \\ C \arrow[r] \arrow[uur, dashed, "h"'] & Y \end{tikzcd}$$

Therefore, $q \circ i_A$ is a trivial cofibration, hence $F(q \circ i_A)$ is a weak equivalence
i.e., $F(q) \circ F(i_A)$ is weak equivalence. Similarly, $F(q) \circ F(i_B)$ is weak equivalence.
Considering $F(p) \circ F(q) \circ F(i_B) = F(id_B)$ is a weak equivalence.
We have $F(p)$ is a weak equivalence.
Hence $F(f) = F(p) \circ (F(q) \circ F(i_A))$ is a weak equivalence. $\square$

1.2 The Homotopy Category

f: Suppose $\mathcal{C}$ is a category with a subcategory of weak equivalences $\mathcal{W}$.
Define the **homotopy "category"** $\mathrm{Ho}\,\mathcal{C}$ as follows.

Step 1: Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and reversals of the arrows of $\mathcal{W}$. That's to say the objects of $F(\mathcal{C}, \mathcal{W}^{-1})$ are the objects of $\mathcal{C}$; and a morphism of $F(\mathcal{C}, \mathcal{W}^{-1})$ is a finite string of composable arrows $(f_1, f_2, \cdots, f_n)$ where $f_i \in \mathrm{Mor}(\mathcal{C})$ or $f_i = g^{-1}$, $g \in \mathrm{Mor}(\mathcal{W})$.
And the empty string at a particular object is the identity map of it.
Composition: $(g_1, g_2, \cdots, g_n) \circ (f_1, \cdots, f_m) = (f_1, f_2, \cdots, f_m, g_1, \cdots, g_n)$

Step 2: $\mathrm{Ho}\,\mathcal{C}$ is the quotient of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations

$$\left\{ \begin{array}{l} 1_A = (1_A) \text{ for all objects } A, \quad (f, g) = g \circ f \text{ for all } f, g \in \mathrm{Mor}(\mathcal{C}) \\ 1_{\mathrm{dom}\,w} = (w, w^{-1}), \quad 1_{\mathrm{codom}\,w} = (w^{-1}, w) \text{ for all } w \in \mathcal{W} \end{array} \right\}$$

nk: Let A, B be two objects of $\mathrm{Ho}\,\mathcal{C}$, then $\mathrm{Mor}_{\mathrm{Ho}\,\mathcal{C}}(A, B)$ may not be a set.

op: ① If $\mathcal{C}$ is a model category, then $\mathrm{Ho}\,D\mathcal{C} = (\mathrm{Ho}\,\mathcal{C})^{op}$
② If $\mathcal{C}, \mathcal{D}$ are model categories, then $\mathrm{Ho}(\mathcal{C} \times \mathcal{D}) \cong \mathrm{Ho}\,\mathcal{C} \times \mathrm{Ho}\,\mathcal{D}$

te: There is a functor $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}\,\mathcal{C}$

objects: $A \longmapsto A$

morphism $f \longmapsto (f)$

$w \mapsto w$ with inverse

emma: Suppose $\mathcal{C}$ is a category with a subcategory $\mathcal{W}$.

(1) If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathcal{W}$ to isomorphisms, then there is a unique functor $HoF : Ho\mathcal{C} \to \mathcal{D}$ s.t $(HoF) \circ \gamma = F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ Ho\mathcal{C} \arrow[ur, dashed, "\exists!\, HoF"'] & \end{tikzcd}$$

where $F(w)$ = isomorphism

(2) Suppose

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\delta"'] & \mathcal{D} \\ \mathcal{E} \arrow[ur, dashed, "\exists!\, F"'] & \end{tikzcd}$$

where $\delta : w \to$ isomorphism.

then $\exists!$ isomorphism $G : Ho\mathcal{C} \to \mathcal{E}$, s.t $G \circ \gamma = \delta$

i.e

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\gamma"] \arrow[ddr, "\delta"'] & D \\ & Ho\mathcal{C} \arrow[u, dashed, "\exists!"'] \arrow[d, dashed, "G\ \exists!"] \\ & \mathcal{E} \arrow[uu, dashed, bend right, "\exists!"'] \end{tikzcd}$$

(3) The correspondence of part (1) induces an isomorphism of categories between ~~$\mathcal{A}$~~ $\mathcal{A} = \left\langle \begin{array}{l} ob : \text{functor } Ho\mathcal{C} \to \mathcal{D} \\ Mor : \text{Natural transformation} \end{array} \right\rangle$ and $\mathcal{B} = \left\langle \begin{array}{l} ob : \text{functor } \mathcal{C} \to \mathcal{D} ;\ w \in \mathcal{W} \to \text{isomo} \\ Mor : \text{Natural transformation} \end{array} \right\rangle$

Pf: (1) Define HoF as

$$HoF(A) = F(A)$$
$$HoF(f) = F(f)$$
$$HoF(w^{-1}) = F(w)^{-1}$$

(2)

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & Ho\mathcal{C} \\ & \mathcal{E} \arrow[u, dashed] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{E} \\ & Ho\mathcal{C} \arrow[u, dashed] \end{tikzcd}$$

(3)

$$\begin{tikzcd} A \arrow[d] & F(A) \arrow[r, "\varphi_A"] \arrow[d] & G(A) \arrow[d] \\ B & F(B) \arrow[r, "\varphi_B"] & G(B) \end{tikzcd} \quad \xrightarrow{\mathcal{B} \to \mathcal{A}} \quad \begin{tikzcd} A \arrow[d, "w^{-1}"] & F(A) \arrow[r, "Ho\varphi_A"] \arrow[d, "F(w)^{-1}"] & G(A) \arrow[d, "G(w)^{-1}"] \\ B & F(B) \arrow[r, "Ho\varphi_B"] & G(B) \end{tikzcd}$$

$$Ho\mathcal{C} \xrightarrow{F} \mathcal{D} \quad \xrightarrow{\mathcal{A} \to \mathcal{B}} \quad \mathcal{C} \xrightarrow{\gamma} Ho\mathcal{C} \xrightarrow{F} \mathcal{D}$$

Prop: Suppose $\mathcal{C}$ is a model category. Let $\mathcal{C}_c$ ($\mathcal{C}_f$, $\mathcal{C}_{cf}$) denote the full subcategory of cofibrant (fibrant, cofibrant and fibrant) objects of $\mathcal{C}$. Then the inclusion functors induce equivalences of categories

$$\mathrm{Ho}\,\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}_c \to \mathrm{Ho}\,\mathcal{C} \quad \text{and} \quad \mathrm{Ho}\,\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}_f \to \mathrm{Ho}\,\mathcal{C}.$$

Pf: We only prove $\mathrm{Ho}\,\mathcal{C}_c \to \mathrm{Ho}\,\mathcal{C}$ is an equivalence of categories.

First,

$$\begin{tikzcd} \mathcal{C}_c \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[r, "r"] & \mathrm{Ho}\,\mathcal{C} \\ \mathrm{Ho}\,\mathcal{C}_c \arrow[urr, dashed, "\exists!\ \mathrm{Ho}(i)"'] & & \end{tikzcd}$$

$$\boxed{\mathcal{C}_c \to \mathcal{C} \qquad \text{weak} \to \text{weak}}$$

Second, recall

$$\begin{tikzcd} \text{Initial} \arrow[rr, "f"] \arrow[dr, "Q : \partial(f)"'] & & X \\ & QX \arrow[ur, "p(f) = \text{trivial fibration} = q"'] & \end{tikzcd}$$

QX = cofibrant

$$\begin{tikzcd} QX \arrow[r, "w"] \arrow[d, "\text{weak}"'] & X \arrow[d, "w"] \\ QY \arrow[r, "w"'] & Y \end{tikzcd}$$

In particular, Q preserves weak equivalence (why?), and hence induces a functor $\mathrm{Ho}\,Q : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}_c$

$$\left(\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_c \arrow[r] & \mathrm{Ho}\,\mathcal{C}_c \\ \mathrm{Ho}\,\mathcal{C} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd} \quad (\text{weak} \to \text{weak} \to \text{isomorphism}) \right)$$

Def: Suppose $\mathcal{C}$ is a model category, and $f, g : B \to X$ are two maps in $\mathcal{C}$.

1. A **cylinder object** for B is a factorization of the fold map $\nabla : B \sqcup B \to B$:

$$\begin{tikzcd} B \sqcup B \arrow[rr, "\nabla"] \arrow[dr, "i_0 + i_1 = \text{cofibration}"'] & & B \\ & B' \arrow[ur, "s = \text{weak equivalence}"'] & \end{tikzcd}$$

Is ∇ induced by id_B and id_B?

2. A **path object** for X is a factorization of the diagonal map $X \to X \times X$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "r = \text{weak equivalence}"'] & & X \times X \\ & X' \arrow[ur, "(p_0, p_1) = \text{fibration}"'] & \end{tikzcd}$$

3. A **left homotopy** from f to g is a map $H: B' \to X$ for some cylinder object B' such that $Hi_0 = f$, and $Hi_1 = g$

i.e.,

$$\begin{tikzcd} B \amalg B \arrow[r, "\Delta"] \arrow[dr, "i_0+i_1"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \\ & B' \arrow[u, "s"] \arrow[ur, "H"'] & \end{tikzcd}$$

$f = H \cdot i_0$
$g = H \cdot i_1$

~~4. A right~~

We say f and g are left homotopic, written $f \overset{l}{\sim} g$, if there is a left homotopy from f to g.

4. A **right homotopy** from f to g is a map K as follow

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "K"'] & X \arrow[r] \arrow[dr, "r"] & X \times X \\ & & X' \arrow[u, "(p_0, p_1)"'] \end{tikzcd}$$

~~where~~ s.t $p_0 K = f$, $p_1 K = g$

We say f and g are right homotopic, written $f \overset{r}{\sim} g$

5. f and g are **homotopic**, written $f \sim g$, if $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$.

6. $f: B \to X$ is a **homotopy equivalence** if $\exists\, h: X \to B$ s.t $hf \sim 1_B$, and $fh \sim 1_X$.

2: ① A path object for B in $\mathcal{C}$ is the same thing as a cylinder object for B in $D\mathcal{C}$.

② Similarly, $f \overset{l}{\sim} g$ in $\mathcal{C}$ $\Leftrightarrow$ $f \overset{r}{\sim} g$ in $D\mathcal{C}$.

1.

$$\begin{tikzcd} B \amalg B \arrow[r, "\Delta"] \arrow[dr, "{(i_0, i_1)}"', "\text{cofibration}"] & B \\ & B' \arrow[u, "s"', "\text{weak equivalence}"] \end{tikzcd}$$

and also

$$\begin{tikzcd} B \amalg B \arrow[r, "\Delta"] \arrow[dr, "\partial(\Delta)", "\text{cofibration}"'] & B \\ & B \times I \arrow[u, "p(\Delta)", "\text{trivial fibration}"'] \end{tikzcd}$$

$\underline{B \times I}$ $\parallel$ functorial cylinder

Then we get

$$\begin{tikzcd} B \cup B \arrow[r, "\partial(\Delta)"] \arrow[d, "{(i_0, i_1)}"'] & B \times I \arrow[d, "p(\Delta) = w \cdot f"] \\ B' \arrow[r, "s"', "w"] \arrow[ur, dashed, "\text{weak equi}"] & B \end{tikzcd}$$

Pf: (1) is straightforward.

(2) First, we claim the cylinder object B' can be replaced by B'' s.t $s': B'' \to B$ is a trivial fibration.

Indeed,
$$\begin{tikzcd} B \sqcup B \arrow[r] \arrow[dr, "{(i_0, i_1)}"'] & B \\ & B' \arrow[u, "s"'] \end{tikzcd}$$
is already know

Consider
$$\begin{tikzcd} B' \arrow[rr, "s"] \arrow[dr, "\partial"'] & & B \\ & B'' \arrow[ur, "\beta"'] & \end{tikzcd}$$

then $s = \beta \circ \partial$, β = trivial fibration, ∂ = cofibration, s = weak equivalence.
By 2-out-of-3 axiom, ∂ is a W.E, hence ∂ = trivial cofibration.

Hence
$$\begin{tikzcd} B \sqcup B \arrow[rr] \arrow[dr] \arrow[drr, bend right, "\text{cofibration}"'] & & B \\ & B' \arrow[r] & B'' \arrow[u, "\text{trivial fibration}"'] \end{tikzcd}$$

tells us B'' is a cylinder object

Next, we claim B'' can also give us a left homotopy.

consider
$$\begin{tikzcd} B' \arrow[r, "H"] \arrow[d, "\text{trivial cofibration}"'] & X \arrow[d, "\text{fibration}"] \\ B'' \arrow[r] \arrow[ur, dashed, "H'"] & * \end{tikzcd}$$

Finally. we claim there is a left homotopy of $A \overset{fh}{\underset{gh}{\rightrightarrows}} X$

consider a cylinder
$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[dr, "j"'] & A \\ & A'' \arrow[u, "t"'] \end{tikzcd}$$
need a map $A'' \to X$
enough if we get $A'' \to B''$

and
$$\begin{tikzcd} A \sqcup A \arrow[r, "{(h,h)}"] & B \sqcup B \arrow[rr] \arrow[dr, "i"'] & & B \\ & & B'' \arrow[ur, "s'"'] & \end{tikzcd}$$

hence we get
$$\begin{tikzcd} A \sqcup A \arrow[r, "{i \circ (h,h)}"] \arrow[d, "j"', "\text{cofibration}"] & B'' \arrow[d, "s'", "\text{trivial fibration}"'] \\ A'' \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

(3) Reflexivity and symmetry are always true no matter B is cofibrant or not.

We already have

$$\begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0+i_1"'] & B \arrow[r, shift left, "t"] \arrow[r, shift right, "g"'] & X \\ & B' \arrow[u, "s"] \arrow[ur, dashed, "H"'] & \end{tikzcd} \qquad \begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0'+i_1'"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \\ & B'' \arrow[u, "s'"] \arrow[ur, dashed, "H'"'] & \end{tikzcd}$$

~~Assume we have trivial cofibrations and trivial fibrations.~~

Let C be the pushout in the diagram below (why exists?) ✓ colimit of $\left\{\begin{matrix} B \to B' \\ B \to B'' \end{matrix}\right.$

why i_1' is a trivial cofibration? ✓

$$\begin{tikzcd} B \arrow[r, "i_0"] \arrow[d, "i_1'"'] & B' \arrow[d, "= \text{trivial cofibration}"] \\ B'' \arrow[r] & C \end{tikzcd}$$

↓ since B is cofibrant

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & B \amalg B \arrow[dr] & \\ & & B' \end{tikzcd}$$

We then have $B \amalg B \xrightarrow{j_0+j_1} C \xrightarrow{t} B$, where $j_0 : B \xrightarrow{i_0} B' \to C$, $j_1 : B \xrightarrow{i_1'} B'' \to C$

and t is induced by $B' \xrightarrow{s} B$ and $B'' \xrightarrow{s'} B$

$$\left(\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d, "w"] \arrow[dd, bend left, "w"] \\ B'' \arrow[r] \arrow[dr, bend right] & C \arrow[d, dashed, "t"] \\ & B \end{tikzcd}\right)$$

Notice t is a weak equivalence. ✓

Meanwhile,

$$\begin{tikzcd} B \arrow[r, "i_0"] \arrow[d, "i_1'"'] & B' \arrow[d] \arrow[ddr, bend left, "H"] & \\ B'' \arrow[r] \arrow[drr, bend right, "H'"'] & C \arrow[dr, dashed, "k"] & \\ & & X \end{tikzcd}$$

tells us

$$\begin{tikzcd} B \amalg B \arrow[r] \arrow[dr, "j_0+j_1"'] & B \arrow[r, shift left, "t"] \arrow[r, shift right, "h"'] & X \\ & C \arrow[u, "t"] \arrow[ur, dashed, "K"'] & \end{tikzcd}$$

s.t $k \circ j_0 = t$, $k \circ j_1 = h$.

Notice j_0+j_1 may NOT be a cofibration, so let's do this

$$\begin{tikzcd} B \amalg B \arrow[rr, "j_0+j_1"] \arrow[dr, "\partial"'] & & C \\ & C' \arrow[ur, "\beta"'] & \end{tikzcd}$$

∂ = cofibration

β = trivial fibration

$$\begin{tikzcd} B \amalg B \arrow[rrr, "\nabla"] \arrow[d, "\partial"'] & & & B \arrow[r, shift left, "t"] \arrow[r, shift right, "h"'] & X \\ C' \arrow[r, "\beta"] & C \arrow[urr, "t"] \arrow[rr, dashed] & & K \arrow[ur] & \end{tikzcd}$$

(4) Suppose $h: X \to Y$ is a trivial fibration.

By (1) an (3), we get a map

$$F: \mathrm{Mor}(B,X)/\overset{l}{\sim} \longrightarrow \mathrm{Mor}(B,Y)/\overset{l}{\sim}$$
$$f \longmapsto h \circ f$$

① claim F is surjective.

If we have $g: B \to Y$, then we have the diagram

$$\begin{tikzcd} \phi \arrow[r] \arrow[d, "\text{cofibration}"'] & X \arrow[d, "h = \text{trivial fibration}"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

so $g = hf$ for some $f \in \mathrm{Mor}(B,X)$

② claim F is injective.

If $hf_1 \overset{l}{\sim} hf_2$, we need to prove $f_1 \sim f_2$

~~consider the diagram~~

We already have

$$\begin{tikzcd} B \amalg B \arrow[r] \arrow[dr, "i_0+i_1"'] & B \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, "h"] & Y \\ & B' \arrow[u, "s"] \arrow[rr, dashed, "H"'] & & \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} B \amalg B \arrow[r, "f_1+f_2"] \arrow[d, "co = i_0+i_1"'] & X \arrow[d, "h = \text{tri fib}"] \\ B' \arrow[r, "H"'] \arrow[ur, dashed, "K"] & Y \end{tikzcd}$$

and hence $Ki_0 = f_1$, $Ki_1 = f_2$

Thus $f_1 \overset{l}{\sim} f_2$

Suppose $h: X \to Y$ is a weak equivalence between fibrant objects