

1. Model Categories

1.1 The Definition of a Model Category

Note: Given a category \mathcal{C} , we can form a new category $\text{Map } \mathcal{C}$ whose objects are morphisms and whose morphisms are commutative squares.

Def: Suppose \mathcal{C} is a category.

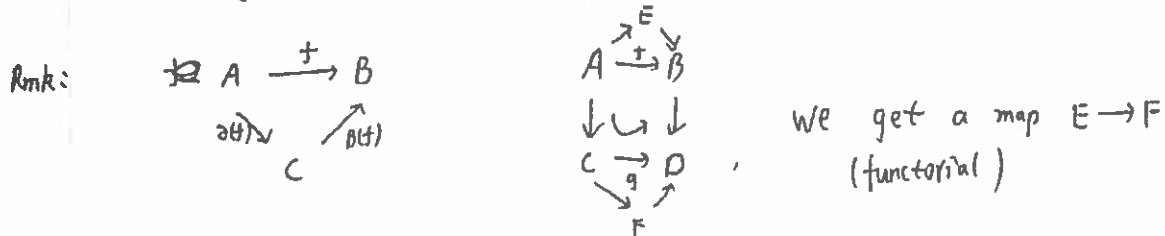
1. A map f is a retract of a map g (both in \mathcal{C}) if f is a retract of g as objects in $\text{Map } \mathcal{C}$.

i.e.

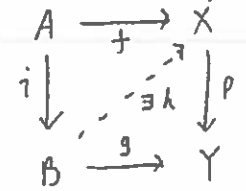
$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

where the horizontal composites are identities.

2. A functorial factorization is an ordered pair (∂, β) of functors $\text{Map } \mathcal{C} \rightarrow \text{Map } \mathcal{C}$, such that $f = \beta(f) \circ \partial(f)$ for all $f \in \text{Map } \mathcal{C}$. In particular, the domain of $\partial(f)$ is the domain of f , $\text{codomain } \partial(f) = \text{domain } \beta(f)$, $\text{codomain } \beta(f) = \text{cod}$.



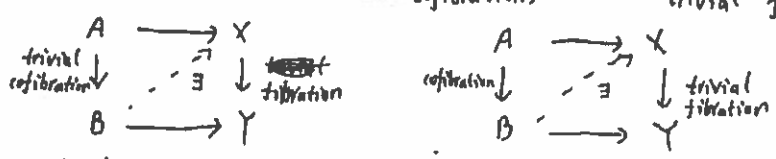
Def: Suppose $i: A \rightarrow B$ and $p: X \rightarrow Y$ are maps in a category \mathcal{C} . Then i has the left lifting property with respect to p and p has the right lifting property with respect to i , if for every commutative diagram of the following form.



Map \mathcal{C} ?
 \uparrow

A model structure on a category \mathcal{C} is three subcategories of \mathcal{C} called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (2-out-of-3) If f and g are morphisms of \mathcal{C} such that gf is defined and two of f, g and gf are weak equivalences, so is the third.
2. (Retracts) If f and g are morphisms of \mathcal{C} such that f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f .
3. (Lifting) Define a map to be a trivial ~~co~~ cofibration if it's both a cofibration and a weak equivalence. Similarly, we can define a trivial fibration. Then trivial cofibrations have the left lifting property with respect to ~~trivial~~ fibrations and ~~trivial~~ cofibrations. trivial fibrations.



4. (Factorization) For any morphism f , $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

A category is said to have all small limits (colimits) if, for every small category \mathcal{J} and any functor $F: \mathcal{J} \rightarrow \mathcal{C}$, a limit (colimit) of F exists in \mathcal{C} .

A model category is a category \mathcal{C} with all small limits and colimits together with a model structure on \mathcal{C} .

- Suppose \mathcal{C} is a category with all small limits and colimits.
- ① weak equivalence \leftrightarrow isomorphism
 - ② fibration, cofibration \leftrightarrow any map in \mathcal{C} .
- ③ $A \xrightarrow{f} B$ $A \xrightarrow{f} B$
 $f = \text{id}_A$ $f = \text{id}_A$ $f = \text{id}_A$ $f = \text{id}_B$

Then this is a model category.

∴ Suppose \mathcal{C} and \mathcal{D} are model categories. Then $\mathcal{C} \times \mathcal{D}$ becomes a model category in an obvious way:

① A map (f, g) is a fibration (cofibration, weak equivalence) if and only if f and g are both fibrations (cofibrations, weak equivalences)

$$\begin{array}{ccc} \textcircled{2} & A \times B \xrightarrow{(f, g)} C \times D & \\ & \downarrow \alpha(f) \times \alpha'(g) & \uparrow \beta(f) \times \beta'(g) \\ & E \times F & \end{array} \qquad \begin{array}{ccc} & A \times B \xrightarrow{(f, g)} C \times D & \\ & \downarrow \alpha(f) \times \alpha'(g) & \uparrow \beta(f) \times \beta'(g) \\ & E \times F & \end{array}$$

∴ If \mathcal{C} is a model category, then \mathcal{C}^{op} is also a model category.

$$\begin{array}{l} \mathcal{C} \longleftrightarrow \mathcal{C}^{op} \\ \text{weak equivalence} \leftrightarrow \text{weak equivalence} \\ \text{fibration} \leftrightarrow \text{cofibration} \\ \text{cofibration} \leftrightarrow \text{fibration} \\ (\partial, \beta) \leftrightarrow \text{new } (\tilde{\gamma}, \tilde{\delta}) \\ (\sigma, \delta) \leftrightarrow \text{new } (\tilde{\alpha}, \tilde{\beta}) \end{array}$$

This is the dual category of \mathcal{C} , denoted by $D\mathcal{C}$, and $D^2\mathcal{C} = \mathcal{C}$.

★ Hence every theorem about model categories has a dual theorem.

k: If \mathcal{C} is a model category, then it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of empty diagram.

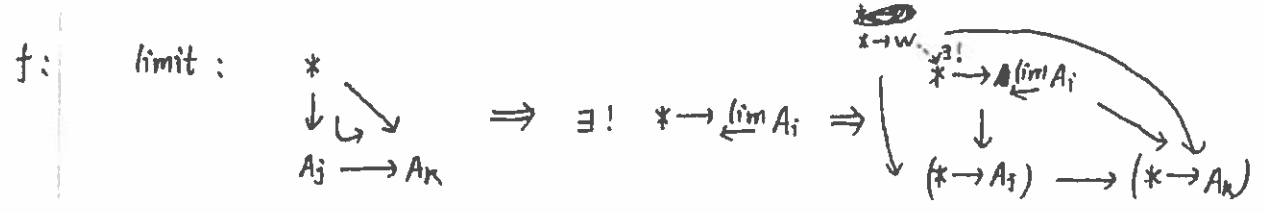
∴ Let \mathcal{C} be a model category. An object is ~~a fibrant~~ fibrant if the map from it to the terminal object is a fibration.
An object is cofibrant if the map from initial object to it is a cofibration.
We can a model category pointed if the map from initial to terminal is an isomorphism.

∴ Given a ~~category~~ model category \mathcal{C} , define the category \mathcal{C}_* to be the category under the terminal object, i.e., induced by k^*

$$\begin{array}{ccc} \text{object: } * & \xrightarrow{v} & A \\ \downarrow \text{id} & \lrcorner & \downarrow t \quad (f: \text{morphism}) \\ & & \cdot \quad \cdot \quad \cdot \end{array}$$

indexed by J

\mathcal{C}_* has arbitrary limits and colimits: $\varprojlim (* \rightarrow A_i) = (* \rightarrow \varprojlim A_i)$
 $\varinjlim (* \rightarrow A_i) = * \rightarrow \varinjlim A_i$



? colimit: Add an extra initial object $*$ to \mathcal{J} , we get \mathcal{J} .
 So $\exists u: * \rightarrow \varinjlim A_i$ (in \mathcal{J}), hence $\exists v: * \rightarrow \varinjlim A_i$ (in \mathcal{C}_*)

And similarly to the limit, we can prove $(* \rightarrow \varinjlim A_i)$ is $\varinjlim (* \rightarrow A_i)$

\mathcal{C}_* is a pointed category, and $*$ is its zero element.

In \mathcal{C}_* , $v: * \rightarrow X$ is an object, and we consider it as an object X along with a basepoint v .

There is an obvious functor $\mathcal{C} \rightarrow \mathcal{C}_*$, $X \mapsto (* \rightarrow X \sqcup *)$

The coproduct of $* \rightarrow X$ and $* \rightarrow Y$ in \mathcal{C}_* is

$$(* \rightarrow X \sqcup Y) / \begin{matrix} (* \rightarrow X \rightarrow X \sqcup Y) \\ (* \rightarrow Y \rightarrow X \sqcup Y) \end{matrix}$$

Suppose \mathcal{C} is a model category. Define a map f in \mathcal{C}_* to be a cofibration (fibration, weak equivalence) if and only if $U(f)$ is a cofibration (fibration, weak equivalence) in \mathcal{C} . Then \mathcal{C}_* is a model category.

Check all the ~~conditions~~ ^{axioms} in the definition.

1. We can replace $*$ by A , to get $h^A: X \rightarrow \text{Mor}(A, X)$ and a category under A .
2. Dually, we can get $h_A: X \rightarrow \text{Mor}(X, A)$.

Let \mathcal{C} be a model category with initial object X .

$$\begin{array}{ccc} \text{Initial} \xrightarrow{f} X & & X \xrightarrow{g} * \\ \text{alt} \searrow \text{alt} & \nearrow \beta = \beta(\text{alt}) & \text{Dually, } \begin{array}{ccc} X & \xrightarrow{g} & * \\ \text{alt} \searrow \text{alt} & \nearrow \beta = \beta(\text{alt}) & \\ \text{RX} & & \end{array} \end{array} \quad \left(\text{where } X \text{ is the terminal object} \right)$$

Indeed, ∂ is a functor, and $\partial(\text{id}_X)$ is a cofibration, hence $QX = \partial(X)$ is cofibrant. We refer to Q as the cofibrant replacement functor of \mathcal{C} .
(Why is it a functor!) \checkmark

Dually, we say R is the fibrant replacement functor of \mathcal{C} .

(The Retract Argument) Suppose we have a factorization $f = p \circ i$ in a category \mathcal{C} , and suppose f has the left lifting property with respect to p . Then f is a retract of i . Dually, if f has the right lifting property with respect to i , then f is a retract of p .

$$\begin{array}{ccc} A \xrightarrow{f} B & \text{and} & A \xrightarrow{i} C \\ \downarrow \text{alt} \nearrow \text{alt} & & \downarrow \text{alt} \nearrow \text{alt} \\ \text{C} & & \text{B} \end{array} \quad \text{hence} \quad \begin{array}{ccc} A \xrightarrow{\text{id}} A & \xrightarrow{\text{id}} & A \\ \downarrow f & \downarrow i & \downarrow p \\ B \xrightarrow{h} C & \xrightarrow{p} & B \end{array}$$

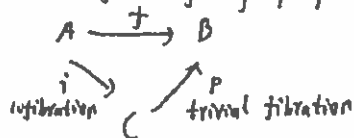
Similar to the duality.

Suppose \mathcal{C} is a model category. Then a map is a cofibration (or trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, fibration (or trivial fibration) \Leftrightarrow it has the right lifting property with respect to all trivial cofibrations (cofibrations).

\Rightarrow : \checkmark

\Leftarrow : Suppose f has left lifting property with respect to all trivial fibrations.

Consider

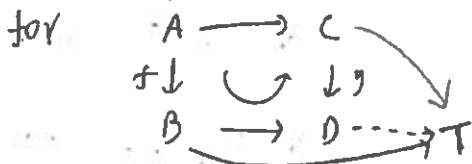


so f is a retract of i .

Hence f is a cofibration.

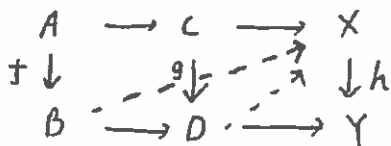
Other situations are similar.

Suppose \mathcal{C} is a model category. Then cofibrations (trivial cofibrations) are closed under pushouts, i.e.,



if f is a cofibration, then g is a cofibration.

Dually. If g is a fibration (trivial fibration), then so is f (for pullback).



ma:

(Ken Brown's Lemma) Suppose \mathcal{C} is a model category and \mathcal{D} is a category with a subcategory of weak equivalences that satisfies the 2-out-of-3 axiom. Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor that takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences.

~~Con~~ Dually, if $\{\text{trivial fibrations between fibrants}\} \xrightarrow{F} \{\text{weak equivalences}\}$, then $\{\text{weak equivalences between fibrant objects}\} \xrightarrow{F} \{\text{weak equivalences}\}$.

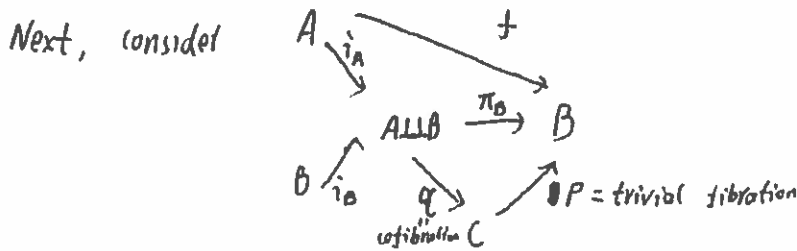
Pf:

~~First, we claim~~ Suppose $f: A \rightarrow B$ is a ~~weakness~~ ^{weak equivalence} between cofibrant objects.

First, we claim $A \rightarrow ALLB$ and $B \rightarrow ALLB$ are cofibrations

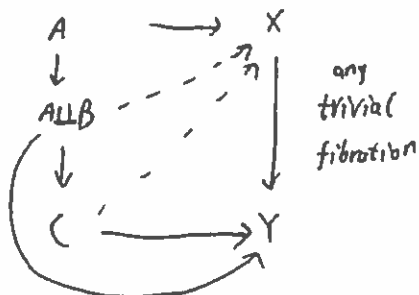
Since $\begin{array}{ccc} 0 & \rightarrow & A \\ \downarrow & & \downarrow \\ B & \rightarrow & A \cup B \end{array}$ is a pushout, and $0 \rightarrow B$ is a cofibration,

we have $A \rightarrow A \cup B$ is a cofibration. Similarly, $B \rightarrow A \cup B$ is a cofibration.



(Claim 1. $q \circ i_A$ and $q \circ i_B$ are weak equivalence
 $p \circ (q \circ i_A) = f$, $p \circ f$ are weak equivalences $\Rightarrow q \circ i_A$ is weak equivalence,
 Similarly, $q \circ i_B$ ✓

(Claim 2. i_A, q are cofibrations, so is $q \circ i_A$, similarly, so is $q \circ i_B$



Therefore, $q \circ i_A$ is a trivial cofibration, hence $F(q \circ i_A)$ is a weak equivalence.
i.e., $F(q) \circ F(i_A)$ is weak equivalence. Similarly, $F(q) \circ F(i_B)$ is weak equivalence.

Considering $F(p) \circ F(q) \circ F(i_A) = F(i_B)$ is a weak equivalence.

We have $F(p)$ is a weak equivalence.

Hence $F(f) = F(p) \circ (F(q) \circ F(i_A))$ is a weak equivalence. \square

1.2 The Homotopy Category

f: Suppose \mathcal{C} is a category with a subcategory of weak equivalences \mathcal{W} . Define the homotopy "category" $\text{Ho}\mathcal{C}$ as follows.

step 1: Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of \mathcal{C} and reversals of the arrows of \mathcal{W} . That's to say the objects of $F(\mathcal{C}, \mathcal{W}^{-1})$ are the objects of \mathcal{C} : and a morphism of $F(\mathcal{C}, \mathcal{W}^{-1})$ is a finite string of composable arrows (f_1, f_2, \dots, f_n) where $f_i \in \text{Mor}(\mathcal{C})$ or $f_i = g^{-1}$, $g \in \text{Mor}(\mathcal{W})$. And the empty string \emptyset at a particular object is the identity map of it.
Composition: $(g_1, g_2, \dots, g_n) \circ (f_1, \dots, f_m) = (f_1, f_2, \dots, f_m, g_1, \dots, g_n)$

Step 2: $\text{Ho}\mathcal{C}$ is the quotient of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations

$$\left\{ \begin{array}{l} |_A = (1_A) \text{ for all objects } A, \quad (f, g) = g \circ f \text{ for all } f, g \in \text{Mor}(\mathcal{C}) \\ |_{\text{dom } W} = (W, W^{-1}), \quad |_{\text{codom } W} = (W^{-1}, W) \text{ for all } W \in \mathcal{W} \end{array} \right\}$$

nk: Let A, B be two objects of $\text{Ho}\mathcal{C}$, then $\text{Mor}_{\text{Ho}\mathcal{C}}(A, B)$ may not be a set.

- op: ① If \mathcal{C} is a model category, then $\text{HoD}\mathcal{C} = (\text{Ho}\mathcal{C})^{\text{op}}$
② If \mathcal{C}, \mathcal{D} are model categories, then $\text{Ho}(\mathcal{C} \times \mathcal{D}) \cong \text{Ho}\mathcal{C} \times \text{Ho}\mathcal{D}$

te: There is a functor $\mathcal{C} \xrightarrow{\gamma} \text{Ho}\mathcal{C}$

objects: $A \mapsto A$

morphism: $f \mapsto (f)$

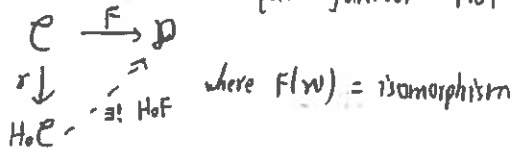
~~$w \mapsto (w^{-1})$~~ $w \mapsto w^{-1}$

$W \mapsto W$ with inverse

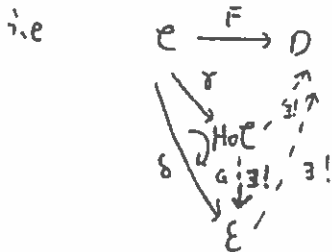
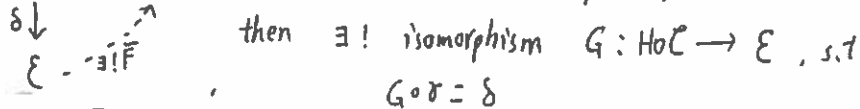
emma:

Suppose \mathcal{C} is a category with a subcategory \mathcal{W} .

(1) If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor that sends maps of \mathcal{W} to isomorphisms, then there is a unique functor $\text{Ho}F: \text{Ho}\mathcal{C} \rightarrow \mathcal{D}$ s.t. $(\text{Ho}F) \circ \gamma = F$

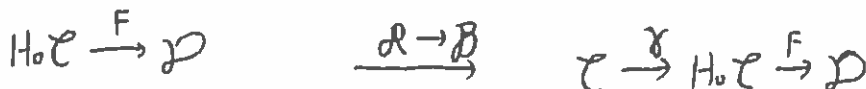
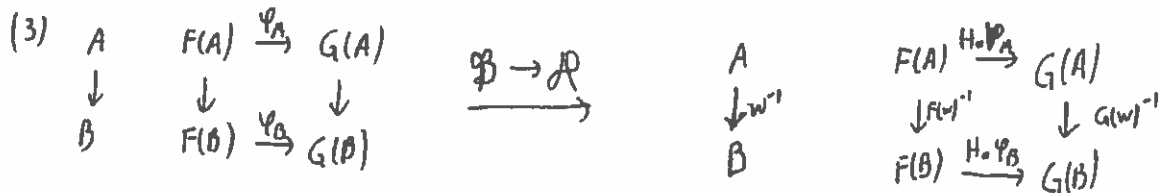
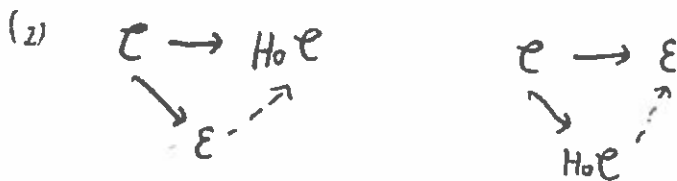


(2) Suppose $\mathcal{C} \xrightarrow{F} \mathcal{D}$ where $\delta: \mathcal{W} \rightarrow \text{isomorphism}$.



(3) The correspondence of part (1) induces an isomorphism of categories between $\mathcal{R} = \left\{ \begin{array}{l} \text{ob: functor } \text{Ho}\mathcal{C} \rightarrow \mathcal{D} \\ \text{Mor: Natural transformation} \end{array} \right\}$ and $\mathcal{B} = \left\{ \begin{array}{l} \text{ob: functor } \mathcal{C} \rightarrow \mathcal{D}; \mathcal{W} \rightarrow \text{isom} \\ \text{Mor: Natural transformation} \end{array} \right\}$

If: (1) Define $\text{Ho}F$ as

$$\begin{aligned} \text{Ho}F(A) &= F(A) \\ \text{Ho}F(f) &= F(f) \\ \text{Ho}F(w^{-1}) &= F(w)^{-1} \end{aligned}$$


Prop:

Suppose \mathcal{C} is a model category. Let \mathcal{C}_c (\mathcal{C}_f , \mathcal{C}_{cf}) denote the full subcategory of cofibrant (fibrant, cofibrant and fibrant) objects of \mathcal{C} .

Then the inclusion functors induce equivalences of categories $\text{Ho}\mathcal{C}_{cf} \rightarrow \text{Ho}\mathcal{C}_c \rightarrow \text{Ho}\mathcal{C}$ and $\text{Ho}\mathcal{C}_{cf} \rightarrow \text{Ho}\mathcal{C}_f \rightarrow \text{Ho}\mathcal{C}$.

Prf:

~~First~~ We only prove $\text{Ho}\mathcal{C}_c \rightarrow \text{Ho}\mathcal{C}$ is an equivalence of categories.

First. $\mathcal{C}_c \xrightarrow{i} \mathcal{C} \xrightarrow{j} \text{Ho}\mathcal{C}$
 $\downarrow \quad \quad \quad \nearrow$
 $\text{Ho}\mathcal{C}_c \quad \quad \quad \text{Ho}(\mathcal{C})$

$\mathcal{C}_c \rightarrow \mathcal{C} \quad \text{weak} \rightarrow \text{weak}$

Second, recall $\text{Initial} \xrightarrow{+} X$

$Q = \partial H \searrow \nearrow p(+)$
 $\nearrow p(+)$ = trivial fibration = q
 $QX = \text{cofibrant}$

$QX \xrightarrow{w} X$
 $\downarrow \quad \downarrow$
 $QY \xrightarrow{w} Y$

In particular, Q preserves weak equivalence (why?), and hence induces a functor $\text{Ho}Q: \text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{C}_c$

$\mathcal{C} \rightarrow \mathcal{C}_c \rightarrow \text{Ho}\mathcal{C}_c$ (weak \rightarrow weak \rightarrow isomorphism)
 $\downarrow \quad \quad \quad \nearrow$
 $\text{Ho}\mathcal{C} \quad \quad \quad \text{Ho}(\mathcal{C})$

Def:

Suppose \mathcal{C} is a model category, and $f, g: B \rightarrow X$ are two maps in \mathcal{C} .

1. A cylinder object for B is a factorization of the fold map

$\nabla: B \amalg B \rightarrow B$:

$B \amalg B \xrightarrow{\nabla} B$
 $\downarrow \quad \downarrow$
 $i_0 + i_1 \quad s$
 $\text{cofibration} \quad \text{weak equivalence}$

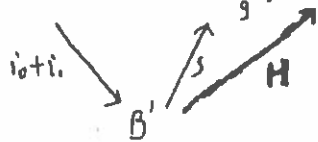
∇ is induced by id_B and id_B ?

2. A path object for X is a factorization of the diagonal map $X \rightarrow X \times X$

$X \rightarrow X \times X$
 $\searrow \quad \nearrow$
 $\delta \quad (P, P)$
 $\text{weak equivalence} \quad \text{fibration}$

3. A left homotopy from f to g is a map $H: B' \rightarrow X$ for some cylinder object B' such that $H \circ i_0 = f$, and $H \circ i_1 = g$

i.e., $B \amalg B \xrightarrow{\Delta} B \xrightarrow{f} X$



$f = H \circ i_0$
 $g = H \circ i_1$

~~4. A right~~

We say f and g are left homotopic, written $f \stackrel{L}{\sim} g$, if there is a left homotopy from f to g .

4. A right homotopy from f to g is a map k as follow

$B \xrightarrow{f} X \rightarrow X \times X$



where s.t. $p \circ k = f$, $p \circ k = g$

We say f and g are right homotopic, written $f \stackrel{R}{\sim} g$

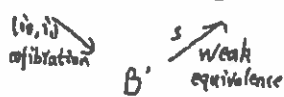
5. f and g are homotopic, written $f \sim g$, if $f \stackrel{L}{\sim} g$ and $f \stackrel{R}{\sim} g$.

6. $f: B \rightarrow X$ is a homotopy equivalence if $\exists h: X \rightarrow B$ s.t. $hf \sim 1_B$, and $fh \sim 1_X$.

① A path object for B in \mathcal{C} is the same thing as a cylinder object for B in DC .

② Similarly, $f \stackrel{L}{\sim} g$ in $\mathcal{C} \iff f \stackrel{R}{\sim} g$ in DC .

1. $B \amalg B \xrightarrow{\Delta} B$



and also,

$B \amalg B \xrightarrow{\Delta} B$



Then we get

$B \amalg B \xrightarrow{\Delta} B$
 $\downarrow (i_0, i_1) \quad \downarrow \text{weak eq.}$
 $B' \xrightarrow{s} B$

functorial cylinder

24:

(1) is straightforward.

(2) First, we claim the cylinder object B' can be replaced by B'' s.t $s': B'' \rightarrow B$ is a trivial fibration.

Indeed, $B \amalg B \rightarrow B$ is already known
 $(i_0, i_1) \rightarrow B' \xrightarrow{s} B$

Consider $B' \xrightarrow{s} B$
 $\downarrow \quad \uparrow \beta$
 $B'' \xrightarrow{s'} B$

then $s = \beta \circ \partial$. $\beta =$ trivial fibration, $\partial =$ cofibration, $s =$ weak equivalence.

By 2-out-of-3 axiom, ∂ is a W.E, hence $\partial =$ trivial cofibration.

Hence $B \amalg B \rightarrow B$
 \downarrow
 $B' \rightarrow B''$ (trivial fibration)
 \downarrow (cofibration)

tells us B'' is a cylinder object

Next, we claim B'' can also give us a left homotopy.

consider $B' \xrightarrow{H} X$
 \downarrow (trivial cofibration) \downarrow (trivial fibration)
 $B'' \xrightarrow{H'} *$

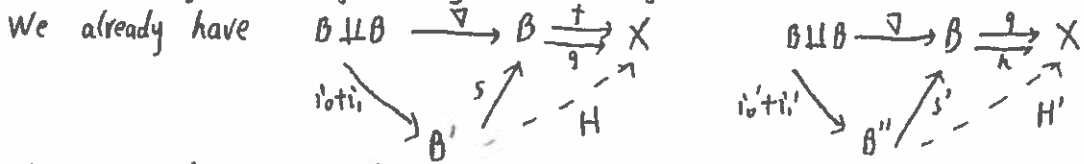
Finally, we claim there is a left homotopy of $A \xrightarrow{f} X$

consider $A \amalg A \rightarrow A$ a cylinder

and $A \amalg A \xrightarrow{(h, h)} B \amalg B \rightarrow B$
 $\downarrow i \quad \uparrow i$
 $A'' \xrightarrow{j} B'' \xrightarrow{s'} B$
 need a map $A'' \rightarrow X$ enough if we get $A'' \rightarrow B''$

hence we get $A \amalg A \xrightarrow{i \circ (h, h)} B''$
 $\downarrow j$ (cofibration) $\downarrow s'$ (trivial fibration)
 $A'' \rightarrow B$

(3) Reflexivity and symmetry are always true no matter B is cofibrant or not.

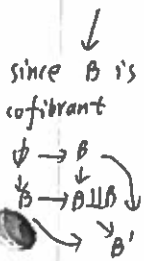


~~Assume we have a trivial cofibration and a trivial fibration.~~

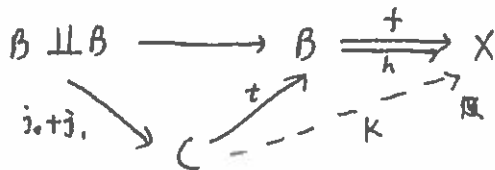
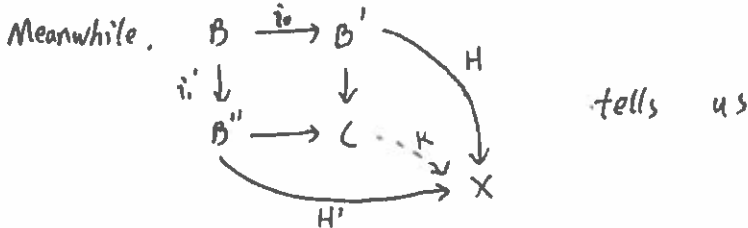
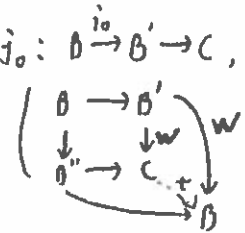
Let C be the pushout in the diagram below (why exists?) ✓



✓ why i_1 is a trivial cofibration?

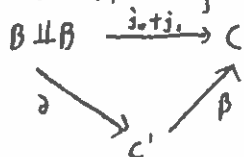


We then have $B \perp B \xrightarrow{j_0 + j_1} C \xrightarrow[t]{\quad} B$, where $j_0: B \xrightarrow{i_0} B' \rightarrow C$, $j_1: B \xrightarrow{i_1} B'' \rightarrow C$ and t is induced by $B' \xrightarrow{s} B$ and $B'' \xrightarrow{s'} B$. Notice t is a weak equivalence. ✓

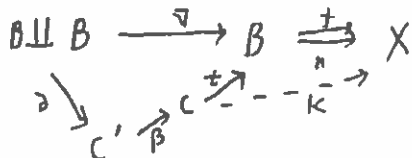


s.t. $k \circ j_0 = t$, $k \circ j_1 = h$.

Notice $j_0 + j_1$ may NOT be a cofibration, so let's do this



α = cofibration
 β = trivial fibration



(4) Suppose $h: X \rightarrow Y$ is a trivial fibration.

By (1) and (3), we get a map

$$F: \text{Mor}(B, X) / \sim \rightarrow \text{Mor}(B, Y) / \sim$$

$$f \mapsto hf$$

(1) claim F is surjective.

If we have $g: B \rightarrow Y$, then we have the diagram

$$\begin{array}{ccc} \phi & \longrightarrow & X \\ \text{trivial fibration} \downarrow & \nearrow & \downarrow h = \text{trivial fibration} \\ B & \xrightarrow{g} & Y \end{array} \quad \text{so } g = hf \text{ for some } f \in \text{Mor}(B, X)$$

(2) claim F is injective.

If $hf_1 \sim hf_2$, we need to prove $f_1 \sim f_2$

~~consider the diagram~~

We already have

$$\begin{array}{ccccc} B \amalg B & \longrightarrow & B & \xrightarrow{f_1} & X & \xrightarrow{h} & Y \\ & \searrow \text{iota}_1 & \nearrow s & \nearrow f_2 & \dashrightarrow & \dashrightarrow & \\ & & B' & & H & & \end{array}$$

Consider the diagram

$$\begin{array}{ccc} B \amalg B & \xrightarrow{f_1 + f_2} & X \\ \text{co} = \text{iota}_1 \downarrow & \dashrightarrow k & \downarrow h = \text{trivial fib} \\ B' & \xrightarrow{H} & Y \end{array} \quad \text{and hence } k \circ \text{iota}_1 = f_1, k \circ \text{iota}_2 = f_2$$

Thus $f_1 \sim f_2$

Suppose $h: X \rightarrow Y$ is a weak equivalence between fibrant objects