Goal: Thm 2.1.19

If $\mathcal{C}$ category w/ all small colimits.

Choose W a subcategory of $\mathcal{C}$ and I, J classes of maps of $\mathcal{C}$.

Then there is a "nice" model structure on $\mathcal{C}$ such that

I generates cofibrations

J generates trivial cofibrations

W weak equivalences.

as long as I, J, W satisfy some technical conditions.

To phrase technical conditions and "generation" (this will be in terms of lifting properties) we define I-inj, I-proj, I-cof, I-fib for some class I of morphisms.

Hard Part: producing functorial factorizations

to construct them, we need the small objects argument, by way of defining proper notion of "small" which will be relative to I-cell = collection of relative I-cell complexes.

I-injective and I-projective

Def^n Let I be any class of maps of category $\mathcal{C}$. A map is

- **I-projective** if it has left lifting property w-r-t maps in I

  I-proj

- map is **I-injective** if it has Right lifting property w-r-t maps of I

  I-inj

- **I-cofibration** if it has Left LP w-r-t maps in I-inj.

  $$I\text{-cof} \\ \| \\ (I\text{-inj})\text{-proj}$$

- **I-fibration** if it has right lifting property w-r-t maps in I-proj.

  $$I\text{-fib} \\ \| \\ (I\text{-proj})\text{-inj}$$

Ex: $\mathcal{C}$ a model category

I = cofibrations in $\mathcal{C}$

then I-inj = trivial fibrations

$$I\text{-cof} = (I\text{-inj})\text{-proj} = I \quad \text{cofibrations}$$

[Dually, I = fibrations then

I-proj = trivial cofibrations

I-fib = $(I\text{-proj})\text{-inj}$ = I fibrations.]

There is a subclass of I-cof which will be important I-cell, which in the case of topological spaces contains the relative CW complexes (and thus CW complexes).

In order to define I-cell = relative I-cell complexes, we need the notion of transfinite composition.

Interlude on Ordinals/Cardinals

Def^n An ordinal is the well-ordered set of all smaller ordinals.

Ex:

$0 = \emptyset$
$1 = \{0\}$
$2 = \{0,1\}$
$3 = \{0,1,2\}$
$\vdots$
$n = \{0,1,\dots,n-1\}$

finite ordinals

ω ordinal containing all finite ordinals.

$\omega = \{0,1,\dots\}$

Note: Each ordinal λ has a successor ordinal $\lambda+1$ which includes all elements of λ and λ itself.

$$\lambda+1 = \{0,1,\dots,\lambda\} = \lambda \cup \{\lambda\}$$

Limit ordinal: Not zero, not a successor. λ limit ordinal if ① $\exists$ ordinal less than λ and ② if $\beta < \lambda$ $\exists$ ordinal γ w/ $\beta < \gamma < \lambda$.

cardinal, transfinite induction.

Every nonzero ordinal is either a successor or a limit ordinal.

Ex $\omega = \{0, 1, 2, \dots\}$ is a limit ordinal

$\omega+1, \omega+2, \omega+3, \dots$ successor ordinals

$\omega+\omega = 2\omega$ limit ordinal

$n\omega$ limit ordinal

Take the union (= supremum) of all $n\omega$ to get $\omega \cdot \omega = \omega^2$ a limit ordinal.

ω^n is a limit ordinal

$\omega^\omega, \ \omega^{\omega^\omega}, \ \omega^{\omega^{\omega^{\cdots}}}$ all limit ordinals. (and countable)

May have seen ordinals in the context of transfinite induction, extending induction on $\mathbb{N}$ to induction on well ordered sets:

For a property P:

① Prove $P(0)$ is true

② Prove that for any ordinal α and ~~any~~ successor $\alpha+1$,

$$P(\alpha) \Rightarrow P(\alpha+1)$$

or

$$P(\beta) \ \forall \beta < \alpha \Rightarrow P(\alpha)$$ (if this, you don't need ③)

③ For any limit ordinal λ

$$[P(\beta) \text{ for } \beta < \lambda] \Rightarrow P(\lambda).$$

② β limit ordinal

$$P(\alpha) \ \forall \alpha < \beta \Rightarrow P(\beta)$$

③ $P(\beta) \Rightarrow P(\beta+1)$

"A property holds for an ordinal as long as it holds for lesser ordinals."

We can view ordinals λ as categories w/ unique maps $\alpha \to \beta$ whenever $\alpha \le \beta$

Here α, β are ordinals which are elements of some larger ordinal λ.

<u>Defn</u> Let $\mathcal{C}$ be a category w/ all small colimits. (colimits indexed over sets). A <u>λ-sequence</u> in $\mathcal{C}$ is a colimit preserving functor $X : \lambda \to \mathcal{C}$

written as $X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$.

so, $X_\beta := X(\beta)$ if $\beta < \lambda$ is an element of λ. (a smaller ordinal)

X preserves colimits, so $\forall$ limit ordinals $\gamma < \lambda$

$$\underset{\beta<\gamma}{\operatorname{colim}} X_\beta = \underset{\beta<\gamma}{\operatorname{colim}} X(\beta) \cong X\big(\underset{\beta<\gamma}{\operatorname{colim}}\, \beta\big) = X(\gamma) = X_\gamma$$

The map $X_0 \to \underset{\beta<\lambda}{\operatorname{colim}} X_\beta$ is the <u>composition</u> of the λ-sequence.

If $\mathcal{D}$ is any collection of morphism in $\mathcal{C}$ and every map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta+1<\lambda$ then $X_0 \to \underset{\beta<\lambda}{\operatorname{colim}} X_\beta$ is the <u>transfinite composition</u> of maps in $\mathcal{D}$.

Def^{n} let α be an ordinal
γ a cardinal

α is <u>γ-filtered</u> if it is

① a limit ordinal

② if $A \subseteq \alpha$ w/ $|A| \leq \gamma$

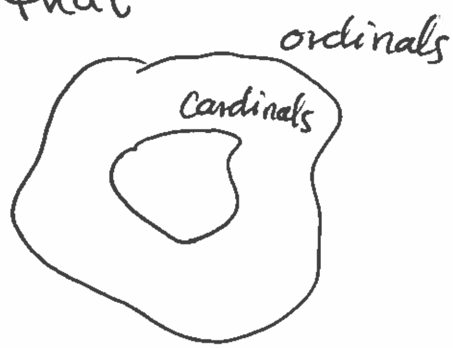
then $\sup A < \alpha$.

Note: ② automatically holds if γ is a finite cardinal

(sup A exists in A if A is finite and ~~$|A| \leq \gamma \Rightarrow \sup A < \gamma$~~ $A \subseteq \alpha \Rightarrow \sup A < \alpha$)

This first!

Cardinals: For a set A, its <u>cardinality</u> $|A|$ is the smallest <u>ordinal</u> such that $\exists$ bijection $|A| \longrightarrow A$.



<u>Ex</u>:
- The ordinal $n = \{0, \dots, n-1\}$ has cardinality n (ordinal) since $\exists$ bijection $n \longrightarrow \{0, \dots, n-1\}$
- $\{1, \dots, n\}$ has cardinality n (ordinal) since $\exists$ bijection $n \longrightarrow \{1, \dots, n\}$.
- ~~the cardinality ω~~ The ordinal ω (considered as a set) has cardinality ω
- The successor $\omega+1$ has cardinality ω

$$\omega \longrightarrow \omega+1$$
$$\{0,1,2,\dots\} \qquad \{0,1,2,\dots,\omega\}$$
$$0 \longmapsto \omega \quad 2 \longmapsto 3$$
$$1 \longmapsto 0 \quad 3 \longmapsto 4 \quad \text{and so on.}$$

Horey uses $\aleph_0$ for cardinality as an ordinal

So there are many more ordinals ~~than~~ than cardinals.

Def^{n} A cardinal is an ordinal α such that $|\alpha| = \alpha$.

<u>Def$^{\underline{n}}$</u> Let $\mathcal{C}$ be category w/ all small colimits, $\mathcal{D}$ collection of morphisms of $\mathcal{C}$ and γ a cardinal.

- A is <u>γ-small relative to $\mathcal{D}$</u> if $\forall$ γ-filtered ordinals λ and all λ-sequences

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots$$

in $\mathcal{D}$ ($\forall \beta$ w/ $\beta+1<\lambda$, $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$)

the set map

$$\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \longrightarrow \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. (Make comment about what this means from Hovey)

- A is <u>small rel. $\mathcal{D}$</u> if it is γ-small rel. $\mathcal{D}$ for some γ.
- A is <u>small</u> if it is small rel. $\mathcal{C}$.
- A is <u>finite rel. $\mathcal{D}$</u> if A is γ-small rel. $\mathcal{D}$ for γ finite
- A is <u>finite</u> if finite rel $\mathcal{C}$.

<u>Moral:</u> A is small if any map from A to a sequence factors through some stage in the sequence.

Ex: ① Any set A is small (it is $|A|$-small rel. Sets)

② A set is finite $\iff$ it is a finite set.

proof: let λ be an $|A|$-filtered ordinal and

let $X : \lambda \to \underline{Sets}$ be a λ-sequence.

let $f : A \to \operatorname{Colim}_{\beta<\lambda} X_\beta$ be a map.

For each $a \in A$ find an index $\beta(a)$ such that

$f(a) \in$ image of $X_{\beta(a)}$ in $\operatorname{Colim}_{\beta<\lambda} X_\beta$

Let $B = \{\beta(a)\}_{a\in A}$. Then $B \subseteq \lambda$ and $|B| \leq |A|$

Since λ is $|A|$-filtered $\Longrightarrow$ $\underbrace{\sup B}_{\gamma} < \lambda$

Then the map $f : A \to \operatorname{Colim} X_\beta$ factors through

$g : A \to X_\gamma$. (This is surjectivity of map on hom sets).

I-cell :

let I be a collection of maps in cat $\mathcal{C}$, and $\mathcal{C}$ contains all small colimits.

A relative I-cell complex is a transfinite composition of pushouts of elements of I

i.e. $f : A \to B$ relative I-cell complex

if $\exists$ ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$

such that
$$\begin{array}{ccc} A & = & X_0 \\ f \downarrow & & \downarrow \\ B & = & \operatorname*{colim}_{\beta < \lambda} X_\lambda \end{array}$$

and for each β w/ $\beta + 1 < \lambda$. $\exists$ pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] & X_{\beta+1} \end{tikzcd} \qquad \text{w/ } g_\beta \in I.$$

let I-cell denote the collection of such morphisms.

An object A is an I-cell complex if $0 \to A$ is a relative I-cell complex.

2.1.10
Lemma : I a class of maps in $\mathcal{C}$ ($\mathcal{C}$ contains all colimits)

Then $I\text{-cell} \subseteq I\text{-cof}$. (see Back)

... Factorizations & model structure

proof: Enough to check that maps in I-cof are closed under pushout and transfinite Comp.

①
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "f'"] \\ B \arrow[r] & B \amalg_A C \end{tikzcd}$$

If $f \in I$-cof, then

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "f'"] & X \arrow[d, "g \in I\text{-fib}"] \\ B \arrow[r] & B \amalg_A C \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

A map $C \to X$ and $B \to X$ gives a map $B \amalg_A C \to X$.

② let $X : \lambda \to \mathcal{C}$ a λ-sequence w/
$X_\beta \to X_{\beta+1} \in I$-cof $\forall \beta$. $(\beta + 1 < \lambda)$

WTS $X_0 \to \operatorname{colim} X_\beta \in I$-cof.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & A \arrow[d, "g"] \\ \operatorname{colim}_{\beta<\lambda} X_\beta \arrow[r] & B \end{tikzcd}$$

a map $\operatorname{colim} X_\beta \to A$ is same as maps $X_\beta \to A$ $\forall \beta$.

we have diagrams.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & A \arrow[d] \\ X_\beta \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

since $X_0 \to X_\beta \in I$-cof. $\forall \beta$.

(From Hirschorn " Quillen Model Category of Top. Spaces")

$X \subseteq Y$ subspace then Y is obtained from X by attaching a cell if there is a pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & Y \end{tikzcd}$$

for some $n \geq 0$.

- A <u>relative cell complex</u> is an inclusion $f: X \to Y$ s.t. Y can be obtained from X by a (possibly infinite) process of attaching cells.

- X is a <u>cell complex</u> if $\emptyset \to X$ is a relative cell complex.

- Every relative CW complex is a relative cell complex

(relative) simplicial complex $\subseteq$ CW complex $\subseteq$ cell complex

But not conversely.

since attaching map in a cell complex is not required to factor through the union of lower dim'l cells.

2.1.14

<u>Theorem</u> (Small Object Argument)

$\mathcal{C}$ category w/ all small colimits and I a set of maps in $\mathcal{C}$. Suppose the domains of maps of I are small relative to I-cell. Then $\exists$ functorial factorization (γ, δ) on $\mathcal{C}$ such that $\forall f$ morphism of $\mathcal{C}$ the map $\gamma(f) \in I$-cell and $\delta(f) \in I$-inj.

<u>(Maybe a proof sketch)</u>:

(1) ~~the~~ For each A ~~the~~ the domain of a map of I. let γ_A be cardinal such that A is γ_A-small rel I-cell. let $\gamma = \bigcup \gamma_A$. Then any domain of a map of I is γ-small rel. I-cell. let λ be a γ-filtered ordinal. (one always exists; suffices to take λ to be the first cardinal larger than γ (Hovey p.29)).

(2) let $f: X \to Y$ be a map in $\mathcal{C}$. we would like to construct a factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "j"'] & & Y \\ & Z_f \arrow[ur, "p"'] & \end{tikzcd}$$

j is in I-cell $\Big($ $\subseteq I$-cof
p is in I-inj $\quad$ $\subseteq$ fibration of model category $\Big)$ cof. gen

In the following manner:

Construct a λ-sequence $Z_0 \to Z_1 \to \cdots \to Z_\beta \to \cdots$ $\beta < \lambda$

such that ① $X = Z_0$

② $Z_f = \operatorname{colim}_{\beta<\lambda} Z_\beta$ and $j : X \to Z_f$

$$\begin{array}{ccc} X & \longrightarrow & Z_f \\ \| & & \| \\ Z_0 & \longrightarrow & \operatorname{colim}_{\beta<\lambda} Z_\beta \end{array}$$

is transfinite composition.

③ $\exists$ map $p_\beta : Z_\beta \to Y$ $\forall \beta$ such that all triangles commute. The induced map

$Z_f = \operatorname{colim}_{\beta<\lambda} Z_\beta \to Y$ is I-injective.

We define all these things using transfinite induction.

<u>0</u>: $Z_0 = X$ and $p_0 : Z_0 \to Y$ is just f.

```tikzcd
X = Z_0 \arrow[r] \arrow[drr, bend right, "p_0"', "f"] & Z_1 \arrow[r] \arrow[dr, "p_1"] & Z_2 \arrow[r] \arrow[d, "p_2"] & \cdots \arrow[r] & Z_\beta \arrow[r] \arrow[dll, bend left, "p_\beta"] & \cdots \\
 & & Y & & &
```

Assume we have defined Z_α, p_α for $\alpha < \beta$ β limit ordinal

Define $Z_\beta = \operatorname{colim}_{\alpha<\beta} Z_\alpha$ and p_β the map induced by p_α.

$Z_\beta \xrightarrow{p_\beta} Y$

Now, we have to define these for the successor ordinal of β.

Consider the set S of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & Z_\beta \arrow[d, "p_\beta"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $g \in I$.

(really S is collection of maps $A_s \xrightarrow{g_s} B_s$ which fit into prescribed diagrams)

For $s \in S$, let $g_s : A_s \to B_s$ be the map in I.

Take coproducts over all such g_s.

$$\begin{tikzcd} \coprod A_s \arrow[r] \arrow[d, "\coprod g_s"'] \arrow[dr, phantom, "\text{pushout}"] & Z_\beta \arrow[d] \\ \coprod B_s \arrow[r] & Z_{\beta+1} \end{tikzcd}$$

$\exists$ map $p_\beta : Z_\beta \to Y$

and $\coprod B_s \to Y$

so we have a diagram

$$\begin{tikzcd} \coprod A_s \arrow[r] \arrow[d, "\coprod g_s"'] & Z_\beta \arrow[d] \arrow[dr, bend left, "p_\beta"] & \\ \coprod B_s \arrow[r] \arrow[rr, bend right] & Z_{\beta+1} \arrow[r, dashed, "p_{\beta+1}"] & Y \end{tikzcd}.$$

~~Remains to show that~~

Induction then gives

a map $Z_f = \operatorname{colim}_{\beta < \lambda} Z_\beta \xrightarrow{p} Y$

It remains to show that this map is in I-inj

(i.e. has R-LP w-r-t I)

The fact that $X \xrightarrow{j} Z_f$ is in I-cell follows

from the fact that I-cell closed under transfinite comp. (Lemma 2.1.12)

and pushouts of coproducts of maps in I are in I-cell (Lemma 2.1.13) why is $Z_f = \operatorname{colim}_{\beta<\lambda} Z_\beta \xrightarrow{p} Y$ in I-inj ?

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & Z_f \arrow[d, "p"] \\ B \arrow[r, "k"'] & Y \end{tikzcd}$$

$g \in I$.

Since domains of maps are γ-small rel I-cell (for some γ), $\exists\ \beta < \gamma$ such that

$$h = A \xrightarrow{h_\beta} Z_\beta \longrightarrow Z^f$$

(factors through some element of colimit.)

But $Z_{\beta+1}$ is a pushout (that's how we built it) and $g \in I$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] \arrow[rr, bend left, "h_\beta"] & \coprod A_s \arrow[r] \arrow[d, "\coprod g_s"] & Z_\beta \arrow[d, "i"] \\ B \arrow[r] \arrow[rr, bend right, "k_\beta"'] & \coprod B_s \arrow[r] & Z_{\beta+1} \end{tikzcd}$$

so that $k_\beta \circ g = i \circ h_\beta$

So we get a map $B \xrightarrow{k_\beta} Z_{\beta+1} \longrightarrow Z^f$

and this is our lift

$\blacksquare$

2.1.3 Cofibrantly generated model Categories

Def$^{\underline{n}}$ A model Category $\mathcal{C}$ is cofibrantly generated if $\exists$ sets I, J such that

① domains of maps of I are small rel I-cell

② ditto for J.

③ fibrations $=$ J-inj

④ trivial fibrations $=$ I-inj

$I =$ generating cofibrations

$J =$ generating trivial ~~f~~ cofibrations

Such a category $\mathcal{C}$ is finitely generated if domains of I and J are finite rel. I-cell for some choice of I, J.

How do we get everything from this data?

Prop$^{\underline{n}}$ 2.1.18 $\mathcal{C}$ cof gen. w/ I, J.

(a) Cof $=$ I-cof.

(d) trivial Cof. $=$ J-cof

<u>Theorem 2.1.19</u> Suppose $\mathcal{C}$ is a category w/ all small limits and colimits. Suppose W subcat. of $\mathcal{C}$ and I, J are sets of maps of $\mathcal{C}$.

$\exists$ a cofibrantly generated model structure w/

I = gen. cof.
J = gen. trivial cof.
W = weak equivalences

$\iff$

1. W 2/3 and closed under retracts
2. domains of I small rel. I-cell
3. ditto for J.
4. $J\text{-cell} \subseteq W \cap I\text{-cof}$
5. $I\text{-inj} \subseteq W \cap J\text{-inj}$
6. either $W \cap I\text{-cof} \subseteq J\text{-cof}$ or $W \cap J\text{-inj} \subseteq I\text{-inj}$

(say some things about this).

$\mathcal{D}$

<u>Lemma 2.1.20</u> Suppose $(F, U, \varphi): \mathcal{C} \to \mathcal{D}$ is an adjunction between model categories. Suppose that $\mathcal{C}$ is cofibrantly generated by I, J.

Then (F, U, φ) is a Quillen ~~equivalence~~ adjunction if and only if $F(f)$ is a cofibration $\forall f \in I$
$F(f)$ is a trivial cofibration $\forall f \in J$.

<u>lemma 2.1.21</u> If $\mathcal{C}$ cof. generated then so is $\mathcal{C}_*$ (attaching basepoint doesn't change things)