1. The Homotopy Category

Corollary 1. *Let $\mathscr{C}$ be a model category. Let $\gamma\colon \mathscr{C}_{cf} \to \mathrm{Ho}\,(\mathscr{C}_{cf})$ and $\delta\colon \mathscr{C}_{cf} \to \mathscr{C}_{cf}/\sim$ be the canonical functors. Then there is a unique isomorphism of categories making the diagram*

$$\begin{tikzcd}
\mathscr{C}_{cf} \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] & & \mathrm{Ho}\,(\mathscr{C}_{cf}) \\
& \mathscr{C}_{cf}/\sim \arrow[ur, dashed, "\exists! j"'] &
\end{tikzcd}$$

commute. Furthermore, j is the identity on objects.

Proof. We show that $\mathscr{C}_{cf}/\sim$ satisfies the same universal property as $\mathrm{Ho}\,(\mathscr{C}_{cf})$. We note that δ takes homotopy equivalences to isomorphisms and so by Proposition 1.2.8, δ also takes weak equivalences to isomorphisms. Let $\mathscr{F} : \mathscr{C}_{cf} \to \mathscr{D}$ be a functor that takes weak equivalences to isomorphisms. Given an object X, take the functorial cylinder object $X \times I$. This guarantees that $X \times I$ is cofibrant and fibrant, since $X \coprod X$ is also cofibrant and X is fibrant. Now note that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] \arrow[rrr, bend left, "\mathrm{id}_X"] \arrow[rrd, shift left, bend right, "\imath_1"] \arrow[rrd, shift right, bend right, "\imath_0"'] & X \coprod X \arrow[rr, "\nabla"] \arrow[rd, "\imath_0 \coprod \imath_1"'] & & X \\
& & X \times I \arrow[ur, "s"'] &
\end{tikzcd}$$

Since s and id_X are weak equivalences we have that $\imath_0$ and $\imath_1$ are both weak equivalences by 2-out-of-3. Hence

$$\mathscr{F}(s) \circ \mathscr{F}(\imath_0) = \mathscr{F}(s \circ \imath_0) = \mathrm{id}_{\mathscr{F}X} = \mathscr{F}(s \circ \imath_1) = \mathscr{F}(s) \circ \mathscr{F}(\imath_1)$$

implies $\mathscr{F}(\imath_0) = \mathscr{F}(\imath_1)$.

Given a homotopy

$$\begin{tikzcd}
X \arrow[r, shift left, "\imath_0"] \arrow[r, shift right, "\imath_1"'] \arrow[dr, shift left, "f_1"] \arrow[dr, shift right, "f_0"'] & X \times I \arrow[d, "H"] \\
& Y
\end{tikzcd}$$

we have

$$\mathscr{F}(f_0) = \mathscr{F}(H \circ \imath_0) = \mathscr{F}(H) \circ \mathscr{F}(\imath_0) = \mathscr{F}(H) \circ \mathscr{F}(\imath_1) = \mathscr{F}(H \circ \imath_1) = \mathscr{F}(f_1)$$

so that $\mathscr{F}$ identifies left homotopic maps. By duality, we see that $\mathscr{F}$ also identifies right homotopic maps. This allows us to define a unique functor $\mathscr{G}\colon \mathscr{C}_{cf}/\sim \to \mathscr{D}$ by $\mathscr{G}\delta X = \mathscr{F}X$ and $\mathscr{G}\delta(f) = \mathscr{F}(f)$. The result now follows by unicity of universal objects. ∎

Theorem 1. *Suppose $\mathscr{C}$ is a model category. Let $\gamma\colon \mathscr{C} \to \mathrm{Ho}\,(\mathscr{C})$ denote the canonical functor, Q the cofibrant replacement functor of $\mathscr{C}$, and R the fibrant replacement functor.*

(i) The inclusion $\mathscr{C}_{cf} \to \mathscr{C}$ induces an equivalence of categories

$$\mathscr{C}_{cf}/\sim \xrightarrow{\sim} \mathrm{Ho}\,(\mathscr{C}_{cf}) \longrightarrow \mathrm{Ho}\,(\mathscr{C})$$

(ii) There are natural isomorphisms

$$\mathscr{C}(QRX, QRY)/\sim \xrightarrow{\sim} \mathrm{Ho}(\mathscr{C})(\gamma X, \gamma Y) \xrightarrow{\sim} \mathscr{C}(RQX, RQY)/\sim$$

In addition, there is a natural isomorphism $\mathrm{Ho}(\mathscr{C})(\gamma X, \gamma Y) \cong \mathscr{C}(QX, RY)/\sim$, *and, if X is cofibrant and Y is fibrant, there is a natural isomorphism* $\mathrm{Ho}(\mathscr{C})(\gamma X, \gamma Y) \cong \mathscr{C}(X, Y)/\sim$. *In particular,* $\mathrm{Ho}(\mathscr{C})$ *is a category without moving to a higher universe.*

(iii) The functor $\gamma\colon \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ identifies left or right homotopic maps.

(iv) If $f\colon A \longrightarrow B$ is a map in $\mathscr{C}$ such that γf is an isomorphism in $\mathrm{Ho}(\mathscr{C})$, then f is a weak equivalence.

Proof. Part (i) is just the composition of the isomorphism from the Corollary above and the equivalence $\mathrm{Ho}(\mathscr{C}_{cf}) \longrightarrow \mathrm{Ho}(\mathscr{C})$ of Proposition 1.2.3.

The first part of (ii) follows from the equivalences

$$\mathrm{Ho}(\mathscr{C}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathscr{C}_c) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathscr{C}_{cf}) \text{ and } \mathrm{Ho}(\mathscr{C}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathscr{C}_f) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathscr{C}_{cf})$$

The rest follows from Proposition 1.2.5 and the fact that

$$QX \xrightarrow{q_X} X \xrightarrow{r_X} RX$$

is a weak equivalence.

Part (iii) was proved in Corollary 1.2.9.

For part (iv), assume that $f \in \mathscr{C}(X, Y)$ is a morphism such that γf is an isomorphism in $\mathrm{Ho}(\mathscr{C})$. This implies that the image of QRf in $\mathscr{C}_{cf}/\sim$ is an isomorphism, and from Corollary 1.2.7 it follows that QRf is a homotopy equivalence. By Proposition 1.2.8, QRf is a weak equivalence and so the left-hand side of the commutative diagram

$$\begin{tikzcd}
QRX \arrow[r, "q_{RX}"] \arrow[d, "QRf"] & RX \arrow[d, "Rf"] & X \arrow[l, "r_X"'] \arrow[d, "f"] \\
QRY \arrow[r, "q_{RY}"] & RY & Y \arrow[l, "r_Y"']
\end{tikzcd}$$

implies by 2-out-of-3 that Rf is a weak equivalence, and thus the right-hand square implies by 2-out-of-3 that f is a weak equivalence, as desired. ■

2. Adjoints

Lemma 1. *Let $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ be a functor and let $\imath_0\colon \mathscr{D}_0 \to \mathscr{D}$ be a full subcategory of $\mathscr{D}$. If for each object X of $\mathscr{C}$ there exists an object Y of $\mathscr{D}_0$ and an isomorphism $\mathscr{F}X \xrightarrow{\sim} \imath_0(Y)$, then there exists a functor $\mathscr{F}_0\colon \mathscr{C} \to \mathscr{D}_0$ and a natural isomorphism $\theta_0\colon \mathscr{F} \to \imath_0 \circ \mathscr{F}_0$. Moreover, $\mathscr{F}_0$ is unique up to unique isomorphism.*

Proof. For each object X of $\mathscr{C}$, choose an object Y of $\mathscr{D}_0$ such that $\theta_0(X)\colon \mathscr{F}X \cong \imath_0(Y)$ and define $\mathscr{F}_0(X) = Y$. Since $\mathscr{F}$ is a functor we have a morphism

$$\mathscr{F}(X_1, X_2)\colon \mathscr{C}(X_1, X_2) \to \mathscr{D}(\mathscr{F}X_1, \mathscr{F}X_2).$$

for each pair of objects X_1, X_2 of $\mathscr{C}$, and since $\mathscr{D}_0$ is a full subcategory, we also have an isomorphism

$$\imath_0(Y_1, Y_2)\colon \mathscr{D}_0(Y_1, Y_2) \xrightarrow{\sim} \mathscr{D}(\imath_0 Y_1, \imath_0 Y_2)$$

for any pair of objects Y_1, Y_2 of $\mathscr{D}_0$. We thus obtain a morphism

$$\begin{tikzcd}
\mathscr{C}(X_1, X_2) \arrow[rrr, "\mathscr{F}(X_1,X_2)"] \arrow[dddddrrr, bend right, "\mathscr{F}_0(X_1,X_2)"'] & & & \mathscr{D}(\mathscr{F}X_1, \mathscr{F}X_2) \arrow[dd, "h_{\mathscr{F}X_2}(\theta_0(X_1)^{-1})"] \\
& f \arrow[r, mapsto] & \mathscr{F}(f) \arrow[d, mapsto] & \\
& & \mathscr{F}(f) \circ \theta_0(X_1)^{-1} \arrow[d, mapsto] & \mathscr{D}(\imath_0 \circ \mathscr{F}_0 X_1, \mathscr{F}X_2) \arrow[d, "h^{\imath_0 \circ \mathscr{F}_0 X}(\theta_0(X_2))"] \\
& & \theta_0(X_2) \circ \mathscr{F}(f) \circ \theta_0(X_1)^{-1} \arrow[d, mapsto] & \mathscr{D}(\imath_0 \circ \mathscr{F}_0 X_1, \imath_0 \circ \mathscr{F}_0 X_2) \arrow[dd, "\imath_0(\mathscr{F}_0 X_1, \mathscr{F}_0 X_2)^{-1}"] \\
& & \imath_0^{-1}(\theta_0(X_2) \circ \mathscr{F}(f) \circ \theta_0(X_1)^{-1}) & \\
& & & \mathscr{D}_0(\mathscr{F}_0 X_1, \circ \mathscr{F}_0 X_2)
\end{tikzcd}$$

Since all the morphisms involved respect composition, it's clear that this assignment is functorial and by construction makes the isomorphism $\theta_0 \colon \mathscr{F} \to \imath_0 \circ \mathscr{F}_0$ natural.

For unicity, suppose that $\mathscr{G}_0 \colon \mathscr{C} \to \mathscr{D}_0$ is another functor equipped with a natural isomorphism $\varphi_0 \colon \mathscr{F} \to \imath_0 \circ \mathscr{G}_0$. We have by assumption an isomorphism of functors

$$\imath_0 \circ G_0 \xrightarrow{\varphi_0^{-1}} \mathscr{F} \xrightarrow{\theta_0} \imath_0 \mathscr{F}_0.$$

Since $\imath_0$ is fully faithful, for each object X of $\mathscr{C}$ it reflects the isomorphism $\theta_0(X) \circ \varphi_0(X)^{-1}$ giving an isomorphism

$$\imath_0^{-1}(\theta_0(X) \circ \varphi_0(X)^{-1}) = \eta(X) \colon \mathscr{G}_0(X) \longrightarrow \mathscr{F}_0(X).$$

To see that this is natural, we consider the diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & & \mathscr{G}_0(X) \arrow[r, "\eta(X)"] \arrow[d, "\mathscr{G}_0(f)"] & \mathscr{F}_0(X) \arrow[d, "\mathscr{F}_0(f)"] \\
X' & & G_0(X') \arrow[r, "\eta(X')"] & \mathscr{F}_0(X')
\end{tikzcd}$$

and note that from the two naturality squares

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r, "\theta_0(X)"] \arrow[d, "\mathscr{F}(f)"] & \imath_0 \circ \mathscr{F}_0(X) \arrow[d, "\imath_0 \circ \mathscr{F}_0(f)"] \\
\mathscr{F}(X') \arrow[r, "\theta_0(X')"] & \imath_0 \circ \mathscr{F}_0(X')
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathscr{F}(X) \arrow[r, "\varphi_0(X)"] \arrow[d, "\mathscr{F}(f)"] & \imath_0 \circ \mathscr{G}_0(X) \arrow[d, "\imath_0 \circ \mathscr{G}_0(f)"] \\
\mathscr{F}(X') \arrow[r, "\varphi_0(X')"] & \imath_0 \circ \mathscr{G}_0(X')
\end{tikzcd}$$

we obtain

$$\begin{aligned}
\imath_0(\mathscr{F}_0(f)\circ\eta(X)) &= \imath_0\circ\mathscr{F}_0(f))\circ\theta_0(X)\circ\varphi_0(X)^{-1} \\
&= \theta_0(X')\circ\mathscr{F}(f)\circ\theta_0(X)^{-1}\circ\theta_0(X)\circ\varphi_0(X)^{-1} \\
&= \theta_0(X')\circ\mathscr{F}(f)\circ\varphi_0(X)^{-1} \\
&= \theta_0(X')\circ\varphi_0(X')^{-1}\circ\imath_0\circ\mathscr{G}_0(f) \\
&= \imath_0(\eta(X'))\circ\imath_0\circ\mathscr{G}_0(f) \\
&= \imath_0(\eta(X')\circ\mathscr{G}_0(f))
\end{aligned}$$

which implies $\mathscr{F}_0(f)\circ\eta(X)=\eta(X')\circ\mathscr{G}_0(f)$ because $\imath_0$ is fully faithful. ■

Proposition 1. *Consider the two functors $\mathscr{L}\colon\mathscr{C}\to\mathscr{D}$ and $\mathscr{R}\colon\mathscr{D}\to\mathscr{C}$. The following are equivalent:*

(i) for each object X of $\mathscr{C}$ and Y of $\mathscr{D}$, there is an isomorphism

$$\mathscr{D}(\mathscr{L}X,Y)\cong\mathscr{C}(X,\mathscr{R}Y)$$

natural in both X and Y,

(ii) there exist natural transformations

$$\varepsilon\colon\ \mathrm{id}_{\mathscr{C}}\to\mathscr{R}\circ\mathscr{L}\ \text{and}\ \eta\colon\mathscr{L}\circ\mathscr{R}\to\mathrm{id}_{\mathscr{D}}$$

called the unit and counit of adjunction, respectively, that make the diagrams

$$\begin{tikzcd}
\mathscr{L}X \arrow[r, "\mathscr{L}(\varepsilon_X)"] \arrow[dr, "\mathrm{id}_{\mathscr{L}X}"'] & \mathscr{L}\circ\mathscr{R}\circ\mathscr{L}X \arrow[d, "\eta_{\mathscr{L}X}"] \\
& \mathscr{L}X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathscr{R}Y \arrow[r, "\varepsilon_{\mathscr{R}Y}"] \arrow[dr, "\mathrm{id}_{\mathscr{R}Y}"'] & \mathscr{R}\circ\mathscr{L}\circ\mathscr{R}Y \arrow[d, "\mathscr{R}(\eta_Y)"] \\
& \mathscr{R}Y
\end{tikzcd}$$

commute for all objects X of $\mathscr{C}$ and Y of $\mathscr{D}$,

(iii) for every object Y of $\mathscr{D}$, the functor

$$\mathscr{D}(\mathscr{L}(-),Y)\colon\mathscr{C}\to\mathfrak{Set}$$

is representable by $\mathscr{R}Y$.

If any of these conditions holds, we say that $\mathscr{L}$ is left adjoint to $\mathscr{R}$ (symmetrically, $\mathscr{R}$ is right adjoint to $\mathscr{L}$), and write $\mathscr{L}\dashv\mathscr{R}$.

Note that condition (iii) implies that the right adjoint of $\mathscr{L}$ is unique up to unique isomorphism and, symmetrically, the left adjoint of $\mathscr{R}$ is as well.

Proof. (i $\Rightarrow$ ii)
For each object X of $\mathscr{C}$ and Y of $\mathscr{D}$ define ε_X and η_Y to be the images of the identity morphism under the isomorphisms

$$\mathscr{D}(\mathscr{L}X,\mathscr{L}X)\xrightarrow{\sim}\mathscr{C}(X,\mathscr{R}\circ\mathscr{L}X)\ \text{and}\ \mathscr{C}(\mathscr{R}Y,\mathscr{R}Y)\xrightarrow{\sim}\mathscr{D}(\mathscr{L}\circ\mathscr{R}Y,Y),$$

respectively. One sees that these are natural transformations by chasing the identities through the commutative diagrams

$$\begin{tikzcd}
X \arrow[dd, "f"] & \mathscr{D}(\mathscr{L}X, \mathscr{L}X) \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{L}X} \circ \mathscr{L}(f)"] & \mathscr{C}(X, \mathscr{R} \circ \mathscr{L}X) \arrow[d, "h^X(\mathscr{R} \circ \mathscr{L}(f))"] \\
 & \mathscr{D}(\mathscr{L}X, \mathscr{L}X') \arrow[r, "\sim"] & \mathscr{C}(X, \mathscr{R} \circ \mathscr{L}X') \\
X' & \mathscr{D}(\mathscr{L}X', \mathscr{L}X')(\arrow[r, "\sim"] \arrow[u, "h_{\mathscr{L}X'}(\mathscr{L}f)"] & \mathscr{C}(X', \mathscr{R} \circ \mathscr{L}X') \arrow[u, "h_{\mathscr{R} \circ \mathscr{L}X'}(f)"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y \arrow[dd, "g"] & \mathscr{C}(\mathscr{R}Y, \mathscr{R}Y) \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{R}Y}(\mathscr{R}(g))"] & \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y, Y) \arrow[d, "h^{\mathscr{L}\mathscr{R}Y}(g)"] \\
 & \mathscr{C}(\mathscr{R}Y, \mathscr{R}Y') \arrow[r, "\sim"] & \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y, Y') \\
Y' & \mathscr{C}(\mathscr{R}Y', \mathscr{R}Y') \arrow[r, "\sim"] \arrow[u, "h_{\mathscr{R}Y'}(\mathscr{R}g)"] & \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y', Y') \arrow[u, "h_{Y'}(\mathscr{L} \circ \mathscr{R}(g))"]
\end{tikzcd}$$

To establish the commutativity conditions, we chase the identities through the commutative diagrams

$$\begin{tikzcd}
X \arrow[d, "\varepsilon_X"] & \mathscr{C}(\mathscr{R} \circ \mathscr{L}X, \mathscr{R} \circ \mathscr{L}X) \arrow[r, "\sim"] \arrow[d, "h_{\mathscr{R} \circ \mathscr{L}X}(\varepsilon_X)"] & \mathscr{D}(\mathscr{L} \circ \mathscr{R} \circ \mathscr{L}X, \mathscr{L}X) \arrow[d, "h_{\mathscr{L}X}(\mathscr{L}(\varepsilon_X))"] \\
\mathscr{R} \circ \mathscr{L}X & \mathscr{C}(X, \mathscr{R} \circ \mathscr{L}X) \arrow[r, "\sim"] & \mathscr{D}(\mathscr{L}X, \mathscr{L}X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{L} \circ \mathscr{R}Y \arrow[d, "\eta_Y"] & \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y, \mathscr{L} \circ \mathscr{R}Y) \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{L} \circ \mathscr{R}Y}(\eta_Y)"] & \mathscr{C}(\mathscr{L} \circ \mathscr{L} \circ \mathscr{R}Y, \mathscr{L}Y) \arrow[d, "h^{\mathscr{R}Y}(\mathscr{R}(\eta_Y))"] \\
Y & \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y, Y) \arrow[r, "\sim"] & \mathscr{C}(\mathscr{R}Y, \mathscr{R}Y)
\end{tikzcd}$$

and use the definition of ε and η. This establishes (ii).

(ii $\Rightarrow$ iii)

Define the morphisms

$$\mathscr{D}(\mathscr{L}X, Y) \overset{\mathscr{R}(\mathscr{L}X, Y)}{\longrightarrow} \mathscr{C}(\mathscr{R} \circ \mathscr{L}X, \mathscr{R}Y) \overset{h_{\mathscr{R}Y}(\varepsilon_X)}{\longrightarrow} \mathscr{C}(X, \mathscr{R}Y)$$

and

$$\mathscr{C}(X, \mathscr{R}Y) \overset{\mathscr{L}(X, \mathscr{R}Y)}{\longrightarrow} \mathscr{D}(\mathscr{L}X, \mathscr{L} \circ \mathscr{R}Y) \overset{h^{\mathscr{L}X}(\eta_Y)}{\longrightarrow} \mathscr{D}(\mathscr{L}X, Y).$$

Using the naturality diagrams

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[d, "f"] & \mathscr{R}\circ\mathscr{L}X \arrow[d, "\mathscr{R}\circ\mathscr{L}(f)"] \\
\mathscr{R}Y \arrow[r, "\varepsilon_{\mathscr{R}Y}"] & \mathscr{R}\circ\mathscr{L}\circ\mathscr{R}Y
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathscr{L}\circ\mathscr{R}\circ\mathscr{L}X \arrow[r, "\eta_{\mathscr{L}X}"] \arrow[d, "\mathscr{R}\circ\mathscr{L}(g)"] & \mathscr{L}X \arrow[d, "g"] \\
\mathscr{L}\circ\mathscr{R}Y \arrow[r, "\eta_Y"] & Y
\end{tikzcd}$$

we see

$$\begin{aligned}
h_{\mathscr{R}Y}(\varepsilon_X)\circ\mathscr{R}(\mathscr{L}X,Y)\circ h^{\mathscr{L}X}(\eta_Y)\circ\mathscr{L}(X,\mathscr{R}Y)(f) &= \mathscr{R}(\eta_Y\circ\mathscr{L}(f))\circ\varepsilon_X \\
&= \mathscr{R}(\eta_Y)\circ\mathscr{R}\circ\mathscr{L}(f)\circ\varepsilon_X \\
&= \mathscr{R}(\eta_Y)\circ\varepsilon_{\mathscr{R}Y}\circ f \\
&= f
\end{aligned}$$

and

$$\begin{aligned}
h^{\mathscr{L}X}(\eta_Y)\circ\mathscr{L}(X,\mathscr{R}Y)\circ h_{\mathscr{R}Y}(\varepsilon_X)\circ\mathscr{R}(\mathscr{L}X,Y)(g) &= \eta_Y\circ\mathscr{L}(\mathscr{R}(g)\circ\varepsilon_X) \\
&= \eta_Y\circ\mathscr{L}\circ\mathscr{R}(g)\circ\mathscr{L}(\varepsilon_X) \\
&= g\circ\eta_{\mathscr{L}X}\circ\mathscr{L}(\varepsilon_X) \\
&= g
\end{aligned}$$

giving isomorphisms

$$\mathscr{D}(\mathscr{L}X,Y)\cong\mathscr{C}(X,\mathscr{R}Y)$$

for each object X of $\mathscr{C}$.

Naturality in X follows directly from naturality of ε and η. Namely, given a morphism $f\in\mathscr{C}(X',X)$ we have the commutative diagram

$$\begin{tikzcd}
X \arrow[ddd, "f"] & \mathscr{D}(\mathscr{L}X,Y) \arrow[rrr, "\sim"] \arrow[ddd, "h_{\mathscr{R}Y}(f)"] & & & \mathscr{C}(X,\mathscr{R}Y) \arrow[ddd, "h_Y(\mathscr{L}(f))"] \\
 & & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \mathscr{R}(\alpha)\circ\varepsilon_X \arrow[d, mapsto] & \\
 & & \alpha\circ\mathscr{L}(f) \arrow[r, mapsto] & \mathscr{R}(\alpha\circ\mathscr{L}(f))\circ\varepsilon_{X'}=\mathscr{R}(\alpha)\circ\varepsilon_X\circ f & \\
X' & \mathscr{D}(\mathscr{L}X',Y) \arrow[rrr, "\sim"] & & & \mathscr{C}(X',\mathscr{R}Y)
\end{tikzcd}$$

and, similarly, the commutative diagram

$$\begin{tikzcd}
X \arrow[ddd, "f"] & \mathscr{C}(X,\mathscr{R}Y) \arrow[rrr, "\sim"] \arrow[ddd, "h_{\mathscr{R}Y}(f)"] & & & \mathscr{D}(\mathscr{L}X,Y) \arrow[ddd, "h_Y(\mathscr{L}(f))"] \\
 & & \beta \arrow[r, mapsto] \arrow[d, mapsto] & \eta_Y\circ\mathscr{L}(\beta) \arrow[d, mapsto] & \\
 & & \beta\circ f \arrow[r, mapsto] & \eta_Y\circ\mathscr{L}(\beta\circ f)=\eta_y\circ\mathscr{L}(\beta)\circ\mathscr{L}(f) & \\
X' & \mathscr{C}(X',\mathscr{R}Y) \arrow[rrr, "\sim"] & & & \mathscr{D}(\mathscr{L}X,Y)
\end{tikzcd}$$

This establishes the isomorphism of functors.

(iii $\Rightarrow$ i)
First define the functor

$$\mathscr{L}_*\colon \operatorname{Fun}(\mathscr{D}^{\mathrm{op}}, \mathfrak{Set}) \to \operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathfrak{Set})$$

as follows. Given $\mathscr{F}\colon \mathscr{D} \to \mathfrak{Set}$ and an object X of $\mathscr{C}$ define

$$\mathscr{L}_*(\mathscr{F})(X) = \mathscr{F} \circ \mathscr{L}(X).$$

Given a morphism $\eta \in \operatorname{Fun}(\mathscr{D}^{\mathrm{op}}, \mathfrak{Set})(\mathscr{F}_1, \mathscr{F}_2)$ define natural transformation $\mathscr{L}_*(\eta)\colon \mathscr{L}_*(\mathscr{F}_1) \to \mathscr{L}_*(\mathscr{F}_2)$ by

$$\mathscr{L}_*(\eta)_X = \eta_{\mathscr{L}X}.$$

Let $h^{\mathscr{D}}_{-}\colon \mathscr{D} \to \operatorname{Fun}(\mathscr{D}^{\mathrm{op}}, \mathfrak{Set})$ and $h^{\mathscr{C}}_{-}\colon \mathscr{C} \to \operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathfrak{Set})$ denote the Yoneda embeddings, which we note identify $\mathscr{C}$ and $\mathscr{D}$ as full subcategories of $\operatorname{Fun}(\mathscr{D}^{\mathrm{op}}, \mathfrak{Set})$ and $\operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathfrak{Set})$ by the Yoneda Lemma. By assumption, we have for each object Y of $\mathscr{D}$ a representing object $\mathscr{R}Y$ of $\mathscr{C}$ for the functor

$$\mathscr{L}_*(h_Y) = \mathscr{D}(\mathscr{L}(-), Y) \cong \mathscr{C}(-, \mathscr{R}Y) = h^{\mathscr{C}}_{-}(\mathscr{R}Y)$$

Hence by Lemma 1 we obtain a factorization

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "h^{\mathscr{D}}_{-}"] \arrow[dr, dashed, "\exists!\mathscr{R}"'] & \operatorname{Fun}(\mathscr{D}^{\mathrm{op}}, \mathfrak{Set}) \arrow[r, "\mathscr{L}_*"] & \operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathfrak{Set}) \\
& \mathscr{C} \arrow[ur, "h^{\mathscr{C}}_{-}"'] &
\end{tikzcd}$$

by the same argument, giving an isomorphism of functors

$$\mathscr{C}(-, \mathscr{R}(-)) = h^{\mathscr{C}}_{-} \circ \mathscr{R} \cong \mathscr{L}_* \circ h^{\mathscr{D}}_{-} = \mathscr{D}(\mathscr{L}(-), -),$$

natural in Y.

Similarly, define the functor $\mathscr{R}_*\colon \operatorname{Fun}(\mathscr{C}, \mathfrak{Set}) \to \operatorname{Fun}(\mathscr{D}, \mathfrak{Set})$ and note that using the co-Yoneda embeddings $h^{-}_{\mathscr{D}}\colon \mathscr{D} \to \operatorname{Fun}(\mathscr{C}, \mathfrak{Set})$ and $h^{-}_{\mathscr{D}}\colon \mathscr{C} \to \operatorname{Fun}(\mathscr{D}, \mathfrak{Set})$ we obtain a factorization

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "h^{-}_{\mathscr{C}}"] \arrow[dr, dashed, "\exists!\mathscr{L}"'] & \operatorname{Fun}(\mathscr{C}, \mathfrak{Set}) \arrow[r, "\mathscr{R}_*"] & \operatorname{Fun}(\mathscr{D}, \mathfrak{Set}) \\
& \mathscr{D} \arrow[ur, "h^{-}_{\mathscr{D}}"'] &
\end{tikzcd}$$

giving an isomorphism of functors

$$\mathscr{D}(\mathscr{L}(-), -) = h^{-}_{\mathscr{D}} \circ \mathscr{L} \cong \mathscr{R}_* \circ h^{-}_{\mathscr{C}} = \mathscr{C}(-, \mathscr{R}(-)),$$

natural in X. This establishes (i). ■

Proposition 2. *Given an adjunction*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, shift left, "\mathscr{L}"] & \mathscr{D} \arrow[l, shift left, "\mathscr{R}"]
\end{tikzcd}$$

the functor $\mathscr{R}$ commutes with limits and, dually, the functor $\mathscr{L}$ commutes with colimits.

Proof. Let $\mathscr{F} : \mathscr{I} \to \mathscr{D}$ be a functor for which $\lim \mathscr{F}$ exists. For every morphism $\alpha\colon i \to j$ of $\mathscr{I}$ we have a commutative diagram of $\mathscr{D}$

$$\begin{tikzcd}
& \lim \mathscr{F} \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
\mathscr{F}(i) \arrow[rr, "\mathscr{F}(\alpha)"] & & \mathscr{F}(j)
\end{tikzcd}$$

and hence obtain a commutative diagram

$$\begin{tikzcd}
& \mathscr{R}(\lim \mathscr{F}) \arrow[dl, "\mathscr{R}(\lambda_i)"'] \arrow[dr, "\mathscr{R}(\lambda_j)"] & \\
\mathscr{R} \circ \mathscr{F}(i) \arrow[rr, "\mathscr{R}\circ\mathscr{F}(\alpha)"] & & \mathscr{R} \circ \mathscr{F}(j)
\end{tikzcd}$$

of $\mathscr{C}$.

Given any other object Z of $\mathscr{C}$ equipped with morphisms $\zeta_i\colon Z \to \mathscr{F}(i)$ such that $\mathscr{R} \circ \mathscr{F}(\alpha) \circ \zeta_i = \zeta_j$ for all $\alpha \in \mathscr{I}(i,j)$ for all i, j, we obtain by the natural isomorphism

$$\begin{tikzcd}
\mathscr{C}(Z, \mathscr{R} \circ \mathscr{F}(i)) \arrow[r, "\mathscr{L}(Z,\mathscr{R}\circ\mathscr{F}(i))"] & \mathscr{D}(\mathscr{L}Z, \mathscr{L} \circ \mathscr{R} \circ \mathscr{F}(i)) \arrow[r, "h^{\mathscr{L}Z}(\varepsilon_{\mathscr{F}(i)})"] & \mathscr{D}(\mathscr{L}Z, \mathscr{F}(i)) \\
\zeta_i \arrow[r, mapsto] & \mathscr{L}(\zeta_i) \arrow[r, mapsto] & \mathscr{L}(\zeta_i) \circ \varepsilon_{\mathscr{F}(i)}
\end{tikzcd}$$

for all objects i of $\mathscr{I}$. By the naturality diagram

$$\begin{tikzcd}
\mathscr{C}(Z, \mathscr{R} \circ \mathscr{F}(i)) \arrow[r, "\sim"] \arrow[d, "h^Z(\mathscr{R}\circ\mathscr{F}(\alpha))"] & \mathscr{D}(\mathscr{R}Z, \mathscr{F}(i)) \arrow[d, "h^{\mathscr{R}Z}(\mathscr{F}(\alpha))"] \\
\mathscr{C}(Z, \mathscr{R} \circ \mathscr{F}(j)) \arrow[r, "\sim"] & \mathscr{D}(\mathscr{R}Z, \mathscr{F}(j))
\end{tikzcd}$$

we see that for all $\alpha \in \mathscr{I}(i,j)$ over all objects i, j of $\mathscr{I}$

$$\mathscr{F}(\alpha) \circ \mathscr{L}(\zeta_i) \circ \varepsilon_{\mathscr{F}(i)} = \mathscr{L}(\mathscr{R} \circ \mathscr{F}(\alpha) \circ \zeta_i) \circ \varepsilon_{\mathscr{F}(j)} = \mathscr{L}(\zeta_j) \circ \varepsilon_{\mathscr{F}(j)}$$

and hence we obtain a unique morphism $h \in \mathscr{D}(\mathscr{R}Z, \lim \mathscr{F})$ making all diagrams

$$\begin{tikzcd}
& \mathscr{L}Z \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "\mathscr{L}(\zeta_i)\circ\varepsilon_{\mathscr{F}(i)}"'] \arrow[ddr, bend left, "\mathscr{L}(\zeta_j)\circ\varepsilon_{\mathscr{F}(j)}"] & \\
& \lim \mathscr{F} \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
\mathscr{F}(i) \arrow[rr, "\mathscr{F}(\alpha)"] & & \mathscr{F}(j)
\end{tikzcd}$$

indexed over $\alpha \in \mathscr{I}(i,j)$ commute. Pulling this map back along the isomorphism

$$\begin{tikzcd}
\mathscr{D}(\mathscr{L}Z, \lim \mathscr{F}) \arrow[r] & \mathscr{C}(\mathscr{R} \circ \mathscr{L}Z, \mathscr{R} \lim \mathscr{F}) \arrow[r] & \mathscr{C}(Z, \mathscr{R} \lim \mathscr{F}) \\
h \arrow[r, mapsto] & \mathscr{R}(h) \arrow[r, mapsto] & \eta_Z \circ \mathscr{R}(h)
\end{tikzcd}$$

gives a moprhism, which we see makes all diagrams

$$\begin{tikzcd}
 & Z \arrow[d, dashed, "\exists ! \eta_Z \circ \mathscr{R}(h)"] \arrow[ddl, bend right, "\zeta_i"'] \arrow[ddr, bend left, "\zeta_j"] & \\
 & \mathscr{R} \lim \mathscr{F} \arrow[dl, "\mathscr{R}(\lambda_i)"'] \arrow[dr, "\mathscr{R}(\lambda_j)"] & \\
\mathscr{R} \circ \mathscr{F}(i) \arrow[rr, "\mathscr{R} \circ \mathscr{F}(\alpha)"] & & \mathscr{R} \circ \mathscr{F}(j)
\end{tikzcd}$$

commute by chasing h through the triangular prism

$$\begin{tikzcd}
 & & \mathscr{D}(\mathscr{L}Z, \lim \mathscr{F}) \arrow[ddll, "h^{\mathscr{L}Z}(\lambda_i)"'] \arrow[d, "\wr"] \arrow[ddrr, "h^{\mathscr{L}Z}(\lambda_j)"] & & \\
 & & \mathscr{C}(Z, \mathscr{R} \lim \mathscr{F}) \arrow[ddll, "h^{Z}(\mathscr{R}(\lambda_i))"] \arrow[ddrr, "h^{Z}(\mathscr{R}(\lambda_j))"'] & & \\
\mathscr{D}(\mathscr{L}Z, \mathscr{F}(i)) \arrow[rrrr, dashed, "h^{\mathscr{L}Z}(\mathscr{F}(\alpha))"] \arrow[d, "\wr"] & & & & \mathscr{D}(\mathscr{L}Z, \mathscr{F}(j)) \arrow[d, "\wr"] \\
\mathscr{C}(Z, \mathscr{R} \circ \mathscr{F}(i)) \arrow[rrrr, "h^{Z}(\mathscr{R} \circ \mathscr{F}(\alpha))"] & & & & \mathscr{C}(Z, \mathscr{R} \circ \mathscr{F}(j))
\end{tikzcd}$$

which commutes since the left, right, and bottom faces are naturality squares and the rear face commutes by construction of h.

We note that $\eta_Z \circ \mathscr{R}(h)$ is necessarily unique, for if we had some other morphism $h' \in \mathscr{C}(Z, \mathscr{R} \lim \mathscr{F})$ making the relevant diagrams of $\mathscr{C}$ commute, then pushing it along the isomorphism gives $\mathscr{L}(h') \circ \varepsilon_{\lim \mathscr{F}} \in \mathscr{D}(\mathscr{L}Z, \lim \mathscr{F})$ making the relevant diagrams of $\mathscr{D}$ commute, which implies $\mathscr{L}(h') \circ \varepsilon_{\lim \mathscr{F}} = h$ by unicity, and hence $h' = \eta_Z \circ \mathscr{R}(h)$. Therefore $\mathscr{R} \lim \mathscr{F} \cong \lim \mathscr{R} \circ \mathscr{F}$ by unicity of universals. ■

Proposition 3. *Let $\mathscr{C}, \mathscr{D}, \mathscr{D}'$ be categories equipped with adjunctions*

$$\mathscr{C} \underset{\mathscr{R}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathscr{D} \quad \textit{and} \quad \mathscr{D} \underset{\mathscr{R}'}{\overset{\mathscr{L}'}{\rightleftarrows}} \mathscr{D}'$$

Then

$$\mathscr{C} \underset{\mathscr{R} \circ \mathscr{R}'}{\overset{\mathscr{L}' \circ \mathscr{L}}{\rightleftarrows}} \mathscr{D}'$$

is an adjunction.

Proof. The composition of functors is clearly well defined, so we obtain a natural isomorphism

$$\begin{aligned}
\mathscr{D}'(\mathscr{L}' \circ \mathscr{L}(X), Y') &\cong \mathscr{D}(\mathscr{L}(X), \mathscr{R}'Y') \\
&\cong \mathscr{C}(X, \mathscr{R} \circ \mathscr{R}'Y').
\end{aligned}$$

■

Definition 1. Given two categories $\mathscr{C}$ and $\mathscr{D}$, an *equivalence of categories* is a pair of functors $\mathscr{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathscr{G}\colon \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \to \mathscr{G} \circ \mathscr{F}$, $\eta\colon \mathscr{F} \circ \mathscr{G} \to \mathrm{id}_{\mathscr{D}}$.

Proposition 4. *Let $\mathscr{L}\colon \mathscr{C} \to \mathscr{D}$ and $\mathscr{R}\colon \mathscr{C} \to \mathscr{D}$ be adjoint functors with unit $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \to \mathscr{R} \circ \mathscr{L}$ and counit $\eta\colon \mathscr{L} \circ \mathscr{R} \to \mathrm{id}_{\mathscr{D}}$.*

(i) The functor $\mathscr{R}$ is fully faithful if and only if η is an isomorphism,
(ii) The functor $\mathscr{L}$ is fully faithful if and only if ε is an isomorphism
(iii) The following are equivalent:
   (a) $\mathscr{L}$ is an equivalence of categories,
   (b) $\mathscr{R}$ is an equivalence of categories,
   (c) $\mathscr{L}$ and $\mathscr{R}$ are fully faithful. In this case, $\mathscr{L}$ and $\mathscr{R}$ are quasi-inverses of one another, and ε, η are both isomorphism.

Proof. For (i), we observe from the diagram

$$\begin{tikzcd}
\mathscr{D}(Y,Y') \arrow[r, "\mathscr{R}(Y,Y')"] \arrow[dr, "h_{Y'}(\eta_Y)"'] & \mathscr{C}(\mathscr{R}Y,\mathscr{R}Y') \arrow[d, "\wr"] \\
& \mathscr{D}(\mathscr{L}\circ\mathscr{R}Y, Y')
\end{tikzcd}$$

that $\mathscr{R}(Y,Y')$ is an isomorphism if and only if $h_{Y'}(\eta_Y)$ is an isomorphism. Hence $\mathscr{R}$ is fully faithful if and only if $h_{Y'}(\eta_Y)$ is an isomorphism for all objects Y, Y' of $\mathscr{D}$, and we are reduced to showing this is equivalent to η_Y being an isomorphism.

Clearly if η_Y is an isomorphism, then $h_{Y'}(\eta_Y)$ is an isomorphism for all objects Y, Y' of $\mathscr{D}$ with inverse

$$h_{Y'}(\eta_Y^{-1})\colon \mathscr{D}(\mathscr{L}\circ\mathscr{R}Y, Y').$$

Conversely, taking $Y' = Y$ we obtain an inverse to η_Y by applying the the inverse morphism:

$$\eta_Y^{-1} = h_Y(\eta_Y)^{-1}(\eta_Y).$$

For (ii), we apply the same argument, mutatis mutandis, to the diagram

$$\begin{tikzcd}
\mathscr{C}(X,X') \arrow[r, "\mathscr{L}(X,X')"] \arrow[dr, "h^X(\varepsilon_X)"] & \mathscr{D}(\mathscr{L}X,\mathscr{L}X') \arrow[d, "\wr"] \\
& \mathscr{C}(X, \mathscr{R}\circ\mathscr{L}X')
\end{tikzcd}$$

Part (iii) follows from the definition of an equivalence of categories and the unicity of the adjoints. ∎

3. Quillen Adjunctions and Derived Functors

Definition 2. Let $\mathscr{C}$ and $\mathscr{D}$ be model categories.

(1) A functor $\mathscr{L}\colon \mathscr{C} \to \mathscr{D}$ is *left Quillen* if $\mathscr{L}$ is a left adjoint and preserves cofibrations and trivial cofibrations.
(2) A functor $\mathscr{R}\colon \mathscr{D} \to \mathscr{C}$ is *right Quillen* if $\mathscr{R}$ is a right adjoint and preserves fibrations and trivial fibrations.

(3) An adjunction $(\mathscr{L}, \mathscr{R}, \varphi)$, where φ is the natural isomorphism

$$\varphi_{X,Y} \colon \mathscr{D}(\mathscr{L}X, Y) \xrightarrow{\sim} \mathscr{C}(X, \mathscr{R}Y),$$

is called a *Quillen adjunction* if $\mathscr{L}$ is left Quillen.

Remark 1. Hovey always uses η for the unit of adjunction and ε for the counit of adjunction; my notation is exactly the opposite.

Example 1. Let $\mathscr{C}$ be a model category and let I be a set. Equip the product category $\mathscr{C}^I$ with the product model structure. We can view an object of $\mathscr{C}^I$ as a discrete diagram of $\mathscr{C}$ and a morphism of this category as a morphisms of discrete diagrams. For example, in the case where I has two elements, a morphism between two objects, $f = (f_1, f_2) \colon X = (X_1, X_2) \to Y = (Y_1, Y_2)$ is just two morphisms of $\mathscr{C}$:

$$\begin{tikzcd} X_1 \arrow[d, "f_1"] & & X_2 \arrow[d, "f_2"] \\ Y_1 & & Y_2 \end{tikzcd}$$

with no commutativity relations.

As $\mathscr{C}$ has all small limits by assumption, we can define a product functor $\lim_I \colon \mathscr{C}^I \to C$ which takes an object $X = (X_i)_I$ of $\mathscr{C}^I$ to the limit over the discrete diagram,

$$\lim_I X = \lim_I X_i = \prod_{i \in I} X_i.$$

For any morphism of $f \in \mathscr{C}^I(X, Y)$, $\lim_I(f)$ is the unique map induced by the universal property of limit, which is determined by the diagrams

$$\begin{tikzcd} \lim_I X \arrow[r, "\lim_I(f)"] \arrow[d] & \lim_I Y \arrow[d] \\ X_i \arrow[r, "f_i"] & Y_i \end{tikzcd}$$

indexed over I. For example, when I has two elements, the image of a morphism $(f_1, f_2) \colon (X_1, X_2) \to (Y_1, Y_2)$ is just the product map,

$$\begin{tikzcd} X_1 \arrow[d, "f_1"] & X_1 \times X_2 \arrow[l] \arrow[r] \arrow[d, dashed, "\exists! f_1 \times f_2"] & X_2 \arrow[d, "f_2"] \\ Y_1 & Y_1 \times Y_2 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

We can define a a diagonal functor $\Delta \colon \mathscr{C} \to \mathscr{C}^I$ which takes an object of X of $\mathscr{C}$ to the discrete diagram with I copies of X. A morphism $f \in \mathscr{C}(X, X)$ defines a morphism of discrete diagrams by taking f in each component. We note that, by definition, a morphism $f \in \mathscr{C}^I(\Delta X, Y)$ is simply a collection of morphisms $f_i \in \mathscr{C}(X, Y_i)$, giving

$$\mathscr{C}^I(\Delta X, Y) = \prod_{i \in I} \mathscr{C}(X, Y_i)$$

With this observation, it's easy to see that this functor is left adjoint to $\lim_I$ since we can view the universal property of products as a representability statement: The object $\lim_I Y_i = \prod_{i \in I} Y_i$ is the representing object of the functor $\prod_{i\in I}\mathscr{C}(-,Y_i)$, and hence

$$\mathscr{C}\left(-,\lim_I Y\right)=\mathscr{C}\left(-,\prod_{i\in I}Y_i\right)\cong\prod_{i\in I}\mathscr{C}\left(-,Y_i\right)=\mathscr{C}^I\left(-,Y\right)=\mathscr{C}^I\left(\Delta\left(-\right),Y\right).$$

By Proposition 1 we obtain the desired natural isomorphism

$$\mathscr{C}^I\left(\Delta X,Y\right)=\prod_{i\in I}\mathscr{C}\left(\Delta X,Y_i\right)\cong\mathscr{C}\left(\Delta X,\lim_I Y\right).$$

Note that by definition of the product model structure, Δ necessarily preserves cofibrations and weak equivalences, so this adjunction is Quillen.

Lemma 2. *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and suppose $\mathscr{L}\colon\mathscr{C}\to\mathscr{D}$ is left adjoint to $\mathscr{R}\colon\mathscr{D}\to\mathscr{C}$. This adjunction is Quillen if and only if $\mathscr{R}$ is right Quillen.*

Proof. By Lemma 1.1.10 of Hovey it suffices to show that given a fibration (resp. trivial fibration) $p\in\mathscr{D}(Y_1,Y_2)$, then $\mathscr{R}p\in\mathscr{C}(\mathscr{R}Y_1,\mathscr{R}Y_2)$ has the right lifting property with respect to all trivial cofibrations (resp. cofibrations). Let $f\in\mathscr{C}(X_1,X_2)$ be a trivial cofibration (resp. cofibration). To say that $\mathscr{R}p$ has the right lifting property with respect to f is to say that for every commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\varphi"] \arrow[d, "f"] & \mathscr{R}Y_1 \arrow[d, "\mathscr{R}p"] \\ X_2 \arrow[r, "\psi"] & \mathscr{R}Y_2 \end{tikzcd}$$

there is a lift $\ell\colon X_2\to\mathscr{R}Y_1$ such that $\ell\circ f=\varphi$ and $\mathscr{R}p\circ\ell=\psi$. This is equivalent to the morphisms of sets

$$\mathscr{D}(\mathscr{L}X_2,Y_1)\cong\mathscr{C}(X_2,\mathscr{R}Y_1)\overset{h^{X_2}(\mathscr{R}p)}{\longrightarrow}\mathscr{C}(X_2,\mathscr{R}Y_2)\cong\mathscr{D}(\mathscr{L}X_2,Y_2)$$

and

$$\mathscr{D}(\mathscr{L}X_2,Y_1)\cong\mathscr{C}(X_2,\mathscr{R}Y_1)\overset{h_{\mathscr{R}Y_1}(f)}{\longrightarrow}\mathscr{C}(X_1,\mathscr{R}Y_1)\cong\mathscr{D}(\mathscr{L}X_1,Y_1)$$

being surjective. However, by the naturality squares

$$\begin{tikzcd} \mathscr{D}(\mathscr{L}X_2,Y_1) \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{L}X_2}(p)"] & \mathscr{C}(X_2,\mathscr{R}Y_1) \arrow[d, "h^{X_2}(\mathscr{R}p)"] \\ \mathscr{D}(\mathscr{L}X_2,Y_2) \arrow[r, "\sim"] & \mathscr{C}(X_2,\mathscr{R}Y_2) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathscr{D}(\mathscr{L}X_2,Y_1) \arrow[r, "\sim"] \arrow[d, "h_{Y_1}(\mathscr{L}(f))"] & \mathscr{C}(X_2,\mathscr{R}Y_1) \arrow[d, "h_{\mathscr{R}Y_1}(f)"] \\ \mathscr{D}(\mathscr{L}X_1,Y_1) \arrow[r, "\sim"] & \mathscr{C}(X_1,\mathscr{R}Y_1) \end{tikzcd}$$

this is equivalent to $\mathscr{L}(f)$ having the left lifting property with respect to p. This is equivalent to $\mathscr{L}(f)$ being a cofibration (resp. trivial cofibration), which in turn is equivalent to the adjunction being Quillen. ■

Remark 2. (i) If $\mathscr{L}\colon\mathscr{C}\to\mathscr{D}$ and $\mathscr{R}\colon\mathscr{D}\to\mathscr{C}$ is a Quillen adjunction, then, by Ken Brown's Lemma, $\mathscr{L}$ preserves weak equivalences between cofibrant objects and $\mathscr{R}$ preserves weak equivalences between fibrant objects. By abuse of notation, we write $\mathscr{L}\colon\mathscr{C}_c\to\mathscr{D}$ and $\mathscr{R}\colon\mathscr{D}_f\to\mathscr{C}$ for the restrictions, then observe that we obtain

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\gamma_{\mathscr{C}}"] & \mathscr{C}_c \arrow[r, "\mathscr{L}"] \arrow[d, "\gamma_{\mathscr{C}_c}"] & \mathscr{D} \arrow[d, "\gamma_{\mathscr{D}}"] \\ \mathrm{Ho}\left(\mathscr{C}\right) \arrow[r, "\mathrm{Ho}(Q)"] & \mathrm{Ho}\left(\mathscr{C}_c\right) \arrow[r, "\mathrm{Ho}(\mathscr{L})"] \arrow[ur, dashed, "\exists!"] & \mathrm{Ho}\left(\mathscr{D}\right) \end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathscr{D} \arrow[r, "R"] \arrow[d, "\gamma_{\mathscr{D}}"] & \mathscr{D}_f \arrow[r, "\mathscr{R}"] \arrow[d, "\gamma_{\mathscr{D}_f}"] & \mathscr{C} \arrow[d, "\gamma_{\mathscr{C}}"] \\
\mathrm{Ho}(\mathscr{D}) \arrow[r, "\mathrm{Ho}(R)"] & \mathrm{Ho}(\mathscr{D}_f) \arrow[r, "\mathrm{Ho}(\mathscr{R})"] \arrow[ur, dashed, "\exists!"] & \mathrm{Ho}(\mathscr{C})
\end{tikzcd}
$$

by the universal property for the homotopy category.

We also note that given natural transformations $\eta\colon \mathscr{L} \to \mathscr{L}'$, $\nu\colon \mathscr{R} \to \mathscr{R}'$ between Quillen functors, we obtain natural transformations $\mathrm{Ho}(\eta):\ \mathrm{Ho}(\mathscr{L}) \to \mathrm{Ho}(\mathscr{L}')$ and $\mathrm{Ho}(\nu):\ \mathrm{Ho}(\mathscr{R}) \to \mathrm{Ho}(\mathscr{R}')$ defined by $\mathrm{Ho}(\eta)_X = \eta_X$ and $\mathrm{Ho}(\nu)_X = \nu_X$. Since all the functors involved preserve weak equivalences, these are indeed natural by Lemma 1.2.2.

Note that on objects,

$$\mathrm{Ho}(\mathscr{L})(X) = \mathscr{L}(QX) \text{ and } \mathrm{Ho}(\mathscr{R})(Y) = \mathscr{R}(QY).$$

(ii) This construction does not require Quillen functors. We only used the fact that these functors preserve weak equivalences between cofibrant (resp. fibrant) objects.

Definition 3. Let $\mathscr{C}$ and $\mathscr{D}$ be model categories.

(1) If $\mathscr{F}:\mathscr{C} \to \mathscr{D}$ is a left Quillen functor, define the *total left derived functor* $L\mathscr{F}:\ \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ to be the composition

$$\mathrm{Ho}(\mathscr{C}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathscr{C}_c) \xrightarrow{\mathrm{Ho}(\mathscr{F})} \mathrm{Ho}(\mathscr{D})$$

Given a natural transformation $\tau\colon \mathscr{F} \to \mathscr{F}'$ of left Quillen functors, define the *total derived natural transformation* $L\tau$ to be $\mathrm{Ho}(\tau) \circ \mathrm{Ho}(Q)$, so that $(L\tau)_X = \tau_{QX}$.

(2) If $\mathscr{G}\colon \mathscr{D} \to \mathscr{C}$ is a right Quillen functor, define the *total right derived functor* $R\mathscr{G}:\ \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$ to be the composition

$$\mathrm{Ho}(\mathscr{D}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathscr{D}_f) \xrightarrow{\mathrm{Ho}(\mathscr{G})} \mathrm{Ho}(\mathscr{C})$$

Given a natural transformation $\tau\colon \mathscr{G} \to \mathscr{G}'$ of right Quillen functors, define the *total derived natural transformation* $R\tau$ to be $\mathrm{Ho}(\tau) \circ \mathrm{Ho}(R)$, so that $(R\tau)_X = \tau_{RX}$.

Theorem 2. *For every model category, $\mathscr{C}$, there is a natural isomorphism $\alpha\colon L(\mathrm{id}_{\mathscr{C}}) \to \mathrm{id}_{\mathrm{Ho}(\mathscr{C})}$. Also for every pair of left Quillen functors $F\colon \mathscr{C} \to \mathscr{D}$ and $F'\colon \mathscr{D} \to \mathscr{E}$, there is a natural isomorphism $m = m_{\mathscr{F}'\mathscr{F}}\colon L\mathscr{F}' \circ L\mathscr{F} \to L(\mathscr{F}' \circ \mathscr{F})$. These natural isomorphisms satisfy the following properties.*

(1) An associativity coherence diagram is commutative. That is, if $\mathscr{F}:\mathscr{C} \to \mathscr{C}'$, $\mathscr{F}'\colon C' \to \mathscr{C}''$, and $F''\colon \mathscr{C}'' \to \mathscr{C}'''$ are left Quillen functors, then the diagram

$$
\begin{tikzcd}
(L\mathscr{F}'' \circ L\mathscr{F}') \circ L\mathscr{F} \arrow[r, "m_{\mathscr{F}''\mathscr{F}'} \circ L\mathscr{F}"] \arrow[d, equal] & L(\mathscr{F}'' \circ \mathscr{F}') \circ L\mathscr{F} \arrow[r, "m_{(\mathscr{F}'' \circ \mathscr{F}')\mathscr{F}}"] & L((\mathscr{F}'' \circ \mathscr{F}') \circ \mathscr{F}) \arrow[d, equal] \\
L\mathscr{F}'' \circ (L\mathscr{F}' \circ L\mathscr{F}) \arrow[r, "L\mathscr{F}'' \circ m_{\mathscr{F}'\mathscr{F}}"] & L\mathscr{F}'' \circ L(\mathscr{F}' \circ \mathscr{F}) \arrow[r, "m_{\mathscr{F}''(\mathscr{F}' \circ \mathscr{F})}"] & L(\mathscr{F}'' \circ (\mathscr{F}' \circ \mathscr{F}))
\end{tikzcd}
$$

commutes.

(2) *A left unit coherence diagram is commutative. That is, if $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, then the diagram*

$$\begin{tikzcd} L(\mathrm{id}_{\mathscr{D}}) \circ L\mathscr{F} \arrow[r, "m"] \arrow[d, "\alpha \circ L\mathscr{F}"] & L(\mathrm{id}_{\mathscr{D}} \circ \mathscr{F}) \arrow[d, equal] \\ 1_{\mathrm{Ho}(\mathscr{D})} \circ L\mathscr{F} \arrow[r, equal] & L\mathscr{F} \end{tikzcd}$$

commutes.

(3) *A right unit coherence diagram is commutative. That is, if $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, then the diagram*

$$\begin{tikzcd} L\mathscr{F} \circ L(\mathrm{id}_{\mathscr{C}}) \arrow[r, "m"] \arrow[d, "L\mathscr{F} \circ \alpha"] & L(\mathscr{F} \circ \mathrm{id}_{\mathscr{C}}) \arrow[d, equal] \\ L\mathscr{F} \circ \mathrm{id}_{\mathrm{Ho}(\mathscr{C})} \arrow[r, equal] & L\mathscr{F} \end{tikzcd}$$

commutes.

Proof. Let $\imath\colon \mathscr{C}_c \to \mathscr{C}$ be the natural inclusion. We observe from the functorial factorization

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & X \\ & QX \arrow[ur, "q_X"'] & \end{tikzcd}$$

that we obtain a natural transformation $q\colon \imath \circ Q \to \mathrm{id}_{\mathscr{C}}$. Taking the derived natural transformation $\mathrm{Ho}\,(q)$ gives the natural isomorphism

$$\mathrm{id}_{\mathrm{Ho}(\mathscr{C})} \xrightarrow{\sim} \mathrm{Ho}\,(\imath) \circ \mathrm{Ho}\,(Q) \xrightarrow{\mathrm{Ho}(q)} L(\mathrm{id}_{\mathscr{C}})$$

because q_X is a trivial fibration.

Define $m_{\mathscr{F}'\mathscr{F}}$ to be the collection of maps

$$L\mathscr{F}' \circ L\mathscr{F}X = \mathscr{F}'(Q(\mathscr{F}(QX))) \xrightarrow{F'(q_{\mathscr{F}(QX)})} \mathscr{F}'(\mathscr{F}(QX)) = L(\mathscr{F}' \circ \mathscr{F}(X))$$

which is natural in X as a functor on $\mathscr{C}$, for if $f' \in \mathscr{C}(X, X')$ then we have the commutative diagram

$$\begin{tikzcd} QX \arrow[r, "q_X"] \arrow[d, "Qf"] & X \arrow[d, "f"] \\ QX' \arrow[r, "q_{X'}"] & X' \end{tikzcd}$$

which gives rise to the commutative diagram in $\mathscr{D}$

$$\begin{tikzcd} Q(\mathscr{F}(QX)) \arrow[r, "q_{\mathscr{F}(QX)}"] \arrow[d, "Q\mathscr{F}(Qf)"] & \mathscr{F}(QX) \arrow[r, "\mathscr{F}(q_X)"] \arrow[d, "\mathscr{F}(Qf)"] & \mathscr{F}(X) \arrow[d, "\mathscr{F}(f)"] \\ Q(\mathscr{F}(QX')) \arrow[r, "q_{\mathscr{F}(QX')}"] & \mathscr{F}(QX') \arrow[r, "\mathscr{F}(q_{X'})"] & \mathscr{F}(X') \end{tikzcd}$$

and hence a commutative diagram in $\mathscr{E}$

$$\begin{array}{ccccc}
\mathscr{F}'(Q(\mathscr{F}(QX))) & \xrightarrow{\mathscr{F}'(q_{\mathscr{F}(QX)})} & \mathscr{F}'\circ\mathscr{F}(QX) & \xrightarrow{\mathscr{F}'\circ\mathscr{F}(q_X)} & \mathscr{F}'\circ\mathscr{F}(X) \\
\downarrow{\scriptstyle \mathscr{F}'(Q\mathscr{F}(Qf))} & & \downarrow{\scriptstyle \mathscr{F}'\circ\mathscr{F}(Qf)} & & \downarrow{\scriptstyle \mathscr{F}'\circ\mathscr{F}(f)} \\
\mathscr{F}'(Q(\mathscr{F}(QX'))) & \xrightarrow{\mathscr{F}'(q_{\mathscr{F}(QX')})} & \mathscr{F}'\circ\mathscr{F}(QX') & \xrightarrow{\mathscr{F}'\circ\mathscr{F}(q_{X'})} & \mathscr{F}'\circ\mathscr{F}(X')
\end{array}$$

Since all functors involved preserve weak equivalences, $m_{\mathscr{F}'\mathscr{F}}$ is also natural in X as a functor on $\mathrm{Ho}\,(\mathscr{C})$. Moreover, $\mathscr{F}$ preserves cofibrant objects because it preserves cofibrations as a left Quillen functor, hence $\mathscr{F}'$ preserves the weak equivalence $q_{\mathscr{F}(QX)}$ between cofibrant objects and thus it follows that $m_{\mathscr{F}'\mathscr{F}}$ is an isomorphism in $\mathrm{Ho}\,(\mathscr{E})$.

For the associativity coherence diagram, we must show that

$$(\mathscr{F}''\circ\mathscr{F}'(q_{\mathscr{F}QX}))\circ(F''(q_{\mathscr{F}'Q\mathscr{F}QX})) = (F''(q_{\mathscr{F}'\mathscr{F}QX}))\circ((\mathscr{F}''\circ Q\circ\mathscr{F}'(q_{\mathscr{F}QX})).$$

This follows from naturality of q and a construction similar to the one above.

The left unit coherence diagram commutes because by definition

$$m_{\mathrm{id}_{\mathscr{D}}\,\mathscr{F}}(X) = \mathrm{id}_{\mathscr{D}}(q_{\mathscr{F}QX}) = q_{\mathscr{F}QX}$$

and

$$\alpha\circ L\mathscr{F}(X) = \alpha_{\mathscr{F}QX} = q_{\mathscr{F}QX}$$

For the right unit coherence diagram, we have

$$m_{\mathscr{F}\,\mathrm{id}_{\mathscr{C}}}(X) = \mathscr{F}(q_{\mathrm{id}_{\mathscr{C}}(q_{QX})}) = \mathscr{F}(q_{QX})\colon \mathscr{F}(QQX)\to\mathscr{F}(QX)$$

and

$$L\mathscr{F}\circ\alpha(X) = \mathscr{F}(Q(q_X))\colon \mathscr{F}(QQX)\to\mathscr{F}(QX).$$

Given a cofibrant object X we have the commutative diagram

$$\begin{array}{ccc}
QQX & \xrightarrow{q_{QX}} & QX \\
\downarrow{\scriptstyle Q(q_X)} & & \downarrow{\scriptstyle q_X} \\
QX & \xrightarrow{q_X} & X
\end{array}$$

because q is natural. Since q_X is a weak equivalence between cofibrant objects it follow that $\mathscr{F}(q_X)$ is invertible in $\mathrm{Ho}\,(\mathscr{D})$ and hence

$$\mathscr{F}(q_{QX}) = \mathscr{F}Q(q_X).$$

Since every object of $\mathrm{Ho}\,(\mathscr{C})$ is weakly equivalent to a cofibrant object, this completes the proof. ■

Definition 4. Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be categories and let $\mathscr{F},\mathscr{G}\colon\mathscr{C}\to\mathscr{D}$, $\mathscr{F}',\mathscr{G}'\colon\mathscr{D}\to\mathscr{E}$ be functors. Given natural transformations $\eta\colon\mathscr{F}\to\mathscr{G}$ and $\nu\colon\mathscr{F}'\to\mathscr{G}'$, defin the *horizontal composition* $\eta*\nu\colon\mathscr{F}'\circ\mathscr{F}\to\mathscr{G}'\circ\mathscr{G}$ is the natural transformation defined by the collection of morphisms of $\mathscr{E}$

$$\begin{array}{ccc}
\mathscr{F}'\circ\mathscr{F}(X) & \xrightarrow{\mathscr{F}'(\eta_X)} & \mathscr{F}'\circ\mathscr{G}(X) \\
\downarrow{\scriptstyle \nu_{\mathscr{F}X}} & & \downarrow{\scriptstyle \nu_{\mathscr{G}X}} \\
\mathscr{G}'\circ\mathscr{F}(X) & \xrightarrow{\mathscr{G}'(\eta_X)} & \mathscr{G}'\circ\mathscr{G}(X)
\end{array}$$

Lemma 3. *Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be model categories. Let $\mathscr{F}, \mathscr{G} \colon \mathscr{C} \to \mathscr{D}$ and $\mathscr{F}', \mathscr{G}' \colon \mathscr{D} \to \mathscr{E}$ be left Quillen functors. Suppose $\eta \colon \mathscr{F} \to \mathscr{G}$ and $\nu \colon \mathscr{F}' \to \mathscr{G}'$ be natural transformations. If m is the composition isomorphism of the Theorem above, then the diagram*

$$\begin{tikzcd} L\mathscr{F}' \circ L\mathscr{F} \arrow[r, "m"] \arrow[d, "L\eta * L\nu"] & L(\mathscr{F}' \circ \mathscr{F}) \arrow[d, "L(\eta * \nu)"] \\ L\mathscr{G}' \circ L\mathscr{G} \arrow[r, "m"] & L(\mathscr{G}' \circ \mathscr{G}) \end{tikzcd}$$

commutes.

Proof. We unravel the definitions. The morphism

$$L(\eta * \nu) \circ m_X : \mathscr{F}'Q\mathscr{F}QX \to \mathscr{G}'\mathscr{G}QX$$

is given by the composition across the top of the commutative diagram

$$\begin{tikzcd} \mathscr{F}'Q\mathscr{F}QX \arrow[r, "\mathscr{F}'(q_{\mathscr{F}QX})"] \arrow[d, "\mathscr{F}'Q(\eta_{QX})"] & \mathscr{F}'\mathscr{F}QX \arrow[r, "\nu_{\mathscr{F}QX}"] \arrow[d, "\mathscr{F}'(\eta_{QX})"] & \mathscr{G}'\mathscr{F}QX \arrow[d, "\mathscr{G}'(\nu_{QX})"] \\ \mathscr{F}'Q\mathscr{G}QX \arrow[r, "\mathscr{F}'(q_{\mathscr{G}QX})"] \arrow[d, "\nu_{Q\mathscr{G}QX}"] & \mathscr{F}'\mathscr{G}QX \arrow[r, "\nu_{\mathscr{G}QX}"] \arrow[d, "\nu_{\mathscr{G}QX}"] & \mathscr{G}'\mathscr{G}QX \arrow[d, equal] \\ \mathscr{G}'Q\mathscr{G}QX \arrow[r, "\mathscr{G}'(q_{\mathscr{G}QX})"] & \mathscr{G}'\mathscr{G}QX \arrow[r, equal] & \mathscr{G}'\mathscr{G}QX \end{tikzcd}$$

and the morphism $m \circ L\eta * L\nu_X \colon \mathscr{F}'Q\mathscr{F}QX \to \mathscr{G}'\mathscr{G}QX$ is given by the composition across the bottom of the commutative diagram

$$\begin{tikzcd} \mathscr{F}'Q\mathscr{F}(QX) \arrow[r, "\nu_{Q\mathscr{F}QX}"] \arrow[d, "\mathscr{F}'Q(\eta_{QX})"] & \mathscr{G}'Q\mathscr{F}QX \arrow[r, "\mathscr{G}'(q_{\mathscr{F}QX})"] \arrow[d, "\mathscr{G}'Q(\eta_{QX})"] & \mathscr{G}'\mathscr{F}QX \arrow[d, "\mathscr{G}'(\eta_{QX})"] \\ \mathscr{F}'Q\mathscr{G}QX \arrow[r, "\nu_{Q\mathscr{G}QX}"] & \mathscr{G}'Q\mathscr{G}QX \arrow[r, "\mathscr{G}'(q_{\mathscr{G}QX})"] & \mathscr{G}'\mathscr{G}QX \end{tikzcd}$$

Chasing the bottom left side of each diagram gives us the desired equality. ∎

Remark 3. Essentially, this says we have a 2-category (modulo some set theoretic issues...) with 0-cells model categories, 1-cells left Quillen functors, 2-cells the natural transformations and the homotopy category, total derived functor, and total derived natural transformation define a pseudo 2-functor to the 2-category of categories.

Lemma 4. *Let $\mathscr{C}$ be a model category and assume we have homotopic morphisms $f_0 \sim f_1 \in \mathscr{C}(X, Y)$. If Y is fibrant object, then we may always choose a fibrant path object. Dually, if X cofibrant we may always choose a cofibrant cylinder object.*

Proof. Since Y is assumed to be fibrant, the pullback

$$\begin{tikzcd} Y \times Y \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] & Y \arrow[d] \\ Y \arrow[r] & 0 \end{tikzcd}$$

is fibrant, as fibrations are stable under pullback. Choose a path object

$$\begin{tikzcd} Y \arrow[rr, "\Delta"] \arrow[dr, "t"'] & & Y \times Y \\ & Y' \arrow[ur, "p_0 \times p_1"'] & \end{tikzcd}$$

with $p_0 \times p_1$ a fibration, t a weak equivalence, and a homotopy

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[dr, "f_i"'] & Y' \arrow[d, "p_i"] \\ & Y \end{tikzcd}$$

Taking a fibrant replacement, $Y \overset{r_Y}{\to} RY'$, we obtain a lift

$$\begin{tikzcd} Y \arrow[d, "t"] \arrow[dr, "\Delta"] & \\ Y' \arrow[r, "p_0 \times p_1"] \arrow[d, "r_Y"] & Y \times Y \arrow[d] \\ RY' \arrow[r] \arrow[ur, dashed, "\exists q_0 \times q_1"] & 0 \end{tikzcd}$$

because $Y \times Y$ is fibrant and r_Y is a trivial cofibration, making RY' a path object. We also obtain a homotopy $r_Y \circ K$ since the diagrams

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[ddrr, "f_i"'] & Y' \arrow[r, "r_Y"] \arrow[dr, "p_0 \times p_1"] \arrow[ddr, "p_i"'] & RY' \arrow[d, "q_0 \times q_1"] \\ & & Y \times Y \arrow[d, "\pi_i"] \\ & & Y \end{tikzcd}$$

commute for $i = 0, 1$. ∎

Lemma 5. *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories. Given a Quillen adjunction*

$$\mathscr{C} \underset{\mathscr{G}}{\overset{\mathscr{F}}{\rightleftarrows}} \mathscr{D}$$

we obtain a derived adjunction

$$\mathrm{Ho}\,(\mathscr{C}) \underset{R\mathscr{G}}{\overset{L\mathscr{F}}{\rightleftarrows}} \mathrm{Ho}\,(\mathscr{D})$$

Proof. Denote by ε and η the unit and counit of adjunction, respectively, and by $[-,-]_{\mathscr{C}}$, $[-,-]_{\mathscr{D}}$ the morphisms of the homotopy category. Also, recall that the isomorphism of adjunction is given by the morphisms

$$\mathscr{D}(\mathscr{F}X,Y) \xrightarrow{\mathscr{G}(\mathscr{F}X,Y)} \mathscr{C}(\mathscr{G}\mathscr{F}X,\mathscr{G}Y) \xrightarrow{h_{\mathscr{G}Y}(\varepsilon_X)} \mathscr{C}(X,\mathscr{G}Y)$$

and

$$\mathscr{C}(X,\mathscr{G}Y) \xrightarrow{\mathscr{F}(X,\mathscr{G}Y)} \mathscr{D}(\mathscr{F}X,\mathscr{F}\mathscr{G}Y) \xrightarrow{h^{\mathscr{F}X}(\eta_Y)} \mathscr{D}(\mathscr{F}X,Y)$$

We first observe that we have natural isomorphisms

$$[L\mathscr{F}X,Y]_{\mathscr{D}} \cong \mathscr{D}(\mathscr{F}QX,RY)/\sim \text{ and } [X,R\mathscr{G}Y]_{\mathscr{C}} \cong \mathscr{C}(QX,\mathscr{G}RY)/\sim.$$

This reduces the problem to showing that the isomorphism of adjunction both preserves and reflects homotopies between cofibrant objects of $\mathscr{C}$ and fibrant objects of $\mathscr{D}$, for then we can see that for any object X of $\mathscr{C}$ and any object Y of $\mathscr{D}$, the isomorphism of adjunction descends to a well-defined isomorphism

$$\begin{tikzcd}
\mathscr{D}(\mathscr{F}QX,RY) \arrow[r, "\sim"] \arrow[d] & \mathscr{C}(QX,\mathscr{G}RY) \arrow[d] \\
\mathscr{D}(\mathscr{F}QX,RY)/\sim \arrow[r] & \mathscr{C}(QX,\mathscr{G}RY)/\sim
\end{tikzcd}$$

Towards that end, assume that X is a cofibrant object of $\mathscr{C}$ and Y is a fibrant object of $\mathscr{D}$. Given $f_0 \sim f_1 \in \mathscr{D}(\mathscr{F}X,Y)$, choose a homotopy from a fibrant path object.

$$\begin{tikzcd}
Y \arrow[rr, "\Delta"] \arrow[dr, "t"'] & & Y \times Y \\
& Y' \arrow[ur, "p_0 \times p_1"'] &
\end{tikzcd}$$

with $p_0 \times p_1$ a fibration, t a weak equivalence, and a homotopy

$$\begin{tikzcd}
\mathscr{F}X \arrow[r, "K"] \arrow[dr, "f_i"'] & Y' \arrow[d, "p_i"] \\
& Y
\end{tikzcd}$$

Since $\mathscr{G}$ is right Quillen it preserves products, fibrant objects, weak equivalences between fibrant objects, and (trivial) fibrations, hence we obtain a path object of $\mathscr{C}$

$$\begin{tikzcd}
\mathscr{G}Y \arrow[rr, "\mathscr{G}(\Delta)"] \arrow[dr, "\mathscr{G}(t)"'] & & \mathscr{G}(Y \times Y) \cong \mathscr{G}(Y) \times \mathscr{G}(Y) \\
& \mathscr{G}Y' \arrow[ur, "\mathscr{G}(p_0 \times p_1) = \mathscr{G}(p_0) \times \mathscr{G}(p_1)"'] &
\end{tikzcd}$$

and a homotopy

$$\begin{tikzcd}
X \arrow[r, "\mathscr{G}(K)\circ\varepsilon_X"] \arrow[dr, "\mathscr{G}(f_i)\circ\varepsilon_X"'] & \mathscr{G}Y' \arrow[d, "\mathscr{G}(p_i)"] \\
& \mathscr{G}Y
\end{tikzcd}$$

by the commutativity of the naturality square

$$\begin{tikzcd} \mathscr{D}(\mathscr{F}X, Y') \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{F}X}(p_i)"] & \mathscr{C}(X, \mathscr{G}Y') \arrow[d, "h^X(\mathscr{G}(p_i))"] \\ \mathscr{D}(\mathscr{F}X, Y) \arrow[r, "\sim"] & \mathscr{C}(X, \mathscr{G}Y) \end{tikzcd}$$

It's clear that this morphism is surjective and natural, so it remains to show that that it reflects homotopies.

Assume that there is a homotopy $\mathscr{G}(f_0) \circ \varepsilon_X \sim \mathscr{G}(f_1) \circ \varepsilon_X$ in $\mathscr{C}$. By duality, the argument above implies that we may choose a cofibrant cylinder object

$$\begin{tikzcd} X \coprod X \arrow[rr, "\nabla"] \arrow[dr, "\imath_0 \coprod \imath_1"'] & & X \\ & X' \arrow[ur, "s"'] & \end{tikzcd}$$

and a homotopy

$$\begin{tikzcd} X \arrow[r, "\imath_i"] \arrow[dr, "\mathscr{G}(f_i) \circ \varepsilon_X"'] & X' \arrow[d, "H"] \\ & \mathscr{G}Y \end{tikzcd}$$

As $\mathscr{F}$ is left Quillen it preserves cofibrant objects, (trivial) cofibrations, and weak equivalences between cofibrant objects by which we obtain a cylinder object in $\mathscr{D}$

$$\begin{tikzcd} \mathscr{F}(X \coprod X) \cong \mathscr{F}X \coprod \mathscr{F}X \arrow[rr, "\mathscr{F}(\nabla)"] \arrow[dr, "\mathscr{F}(\imath_0 \coprod \imath_1) = \mathscr{F}(\imath_0) \coprod \mathscr{F}(\imath_1)"'] & & \mathscr{F}X \\ & \mathscr{F}X' \arrow[ur, "\mathscr{F}(s)"'] & \end{tikzcd}$$

and a homotopy

$$\begin{tikzcd} \mathscr{F}X \arrow[r, "\mathscr{F}(\imath_i)"] \arrow[dr, "\eta_Y \circ \mathscr{F}(\mathscr{G}(f_i) \circ \varepsilon_X)"'] & \mathscr{F}X' \arrow[d, "\eta_Y \circ \mathscr{F}(H)"] \\ & Y \end{tikzcd}$$

and we note that

$$\eta_Y \circ \mathscr{F} \circ \mathscr{G}(f_i) \circ \mathscr{F}(\varepsilon_X) = f_i$$

implies that we have a homotopy $f_0 \sim f_1$, as desired. Therefore the induced map is injective, hence an isomorphism. ■

4. Quillen Equivalences

Definition 5. A Quillen adjunction $(\mathscr{F}, \mathscr{G}, \varphi)\colon \mathscr{C} \to \mathscr{D}$ is a *Quillen equivalence* if and only if, for all cofibrant X in $\mathscr{C}$ and fibrant Y in $\mathscr{D}$, a map $f\colon \mathscr{F}X \to Y$ is a weak equivalence in $\mathscr{D}$ if and only if $\varphi(f)\colon X \to \mathscr{G}Y$ is a weak equivalence in $\mathscr{C}$.

Remark 4. Note that a Quillen equivalence is *not* always an equivalence of categories. This could be thought of as a weak equivalence of model categories.

Proposition 5. *Let $(\mathscr{F},\mathscr{G},\varphi)\colon \mathscr{C} \to \mathscr{D}$ be a Quillen adjunction with unit $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \to \mathscr{G}\circ\mathscr{F}$ and counit $\eta\colon \mathscr{F}\circ\mathscr{G} \to \mathrm{id}_{\mathscr{D}}$. The following are equivalent:*

(a) $(\mathscr{F},\mathscr{G},\varphi)$ is a Quillen equivalence,

(b) The composition $X \xrightarrow{\varepsilon_X} \mathscr{G}\mathscr{F}X \xrightarrow{\mathscr{G}(r_{\mathscr{F}X})} \mathscr{G}R\mathscr{F}X$ is a weak equivalence for all cofibrant X, and the composition $\mathscr{F}Q\mathscr{G}Y \xrightarrow{\mathscr{F}(q_{\mathscr{G}Y})} \mathscr{F}QY \xrightarrow{\eta_Y} Y$ is a weak equivalence for all fibrant Y,

(c) The derived adjunction is an equivalence of categories.

Proof. (a) $\Rightarrow$ (b)
Assume X is cofibrant in $\mathscr{C}$ and Y is fibrant in $\mathscr{D}$. Note that $r_{\mathscr{F}X}$ and $q_{\mathscr{G}Y}$ are both weak equivalences. From the diagrams

$$\begin{tikzcd}
\mathscr{D}(\mathscr{F}X, R\mathscr{F}X) \arrow[r, "\mathscr{F}(X,R\mathscr{F}X)"] \arrow[rr, bend left, "\varphi"] & \mathscr{C}(\mathscr{G}\mathscr{F}X, \mathscr{G}R\mathscr{F}X) \arrow[r, "h_{\mathscr{G}R\mathscr{F}X}(\varepsilon_X)"] & \mathscr{C}(X, \mathscr{G}R\mathscr{F}X) \\
r_{\mathscr{F}X} \arrow[r, mapsto] & \mathscr{G}(r_{\mathscr{F}X}) \arrow[r, mapsto] & \mathscr{G}(r_{\mathscr{F}X})\circ\varepsilon_X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{C}(Q\mathscr{G}Y, \mathscr{G}Y) \arrow[r] \arrow[rr, bend left, "\varphi^{-1}"] & \mathscr{D}(\mathscr{F}Q\mathscr{G}Y, \mathscr{F}\mathscr{G}Y) \arrow[r] & \mathscr{D}(\mathscr{F}G\mathscr{G}Y, Y) \\
q_{\mathscr{G}Y} \arrow[r, mapsto] & \mathscr{F}(q_{\mathscr{G}Y}) \arrow[r, mapsto] & \eta_Y\circ\mathscr{F}(q_{\mathscr{G}Y})
\end{tikzcd}$$

we see $\varphi(r_{\mathscr{F}X}) = \mathscr{G}(r_{\mathscr{F}X}\circ\varepsilon_X)$ and $\varphi(\eta_Y\circ\mathscr{F}(q_{\mathscr{G}Y})) = q_{\mathscr{G}Y}$ imply $\mathscr{G}(r_{\mathscr{F}X})\circ\varepsilon_X$ and $\eta_Y\circ\mathscr{F}(q_{\mathscr{G}Y})$ are weak equivalences by the definition of a Quillen equivalence.

(b) $\Rightarrow$ (c)
It suffices to show that the unit and counit of the adjunction are both isomorphisms. First assume that X is a cofibrant object of $\mathscr{C}$ and note that $\mathscr{F}X$ is cofibrant in X. Chasing the identity through the diagram

$$\begin{tikzcd}
\mathscr{D}(\mathscr{F}X, \mathscr{F}X) \arrow[r, "\sim"] \arrow[d, "h^{\mathscr{F}X}(r_{\mathscr{F}X})"] & \mathscr{C}(X, \mathscr{G}\mathscr{F}X) \arrow[d, "h^X(\mathscr{G}(r_{\mathscr{F}X}))"] \\
\mathscr{D}(\mathscr{F}X, R\mathscr{F}X) \arrow[r, "\sim"] \arrow[d] & \mathscr{C}(X, \mathscr{G}R\mathscr{F}X) \arrow[d] \\
\mathscr{D}(\mathscr{F}X, R\mathscr{F}X)/\sim \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathscr{C}(X, \mathscr{G}R\mathscr{F}X)/\sim \arrow[d, "\wr"] \\
{[L\mathscr{F}X, L\mathscr{F}X]_{\mathscr{D}}} \arrow[r, "\sim"] & {[X, R\mathscr{G}\circ L\mathscr{F}X]_{\mathscr{C}}}
\end{tikzcd}$$

we see that the image in the bottom right hand corner is the isomorphism

$$X \xrightarrow{\varepsilon_X} \mathscr{G}\mathscr{F}X \xrightarrow{\mathscr{G}(r_{\mathscr{F}X})} \mathscr{G}R\mathscr{F}X$$

and thus for arbitrary X we obtain the isomorphism

$$X \xrightarrow{q_X^{-1}} QX \xrightarrow{\varepsilon_{QX}} \mathscr{G}\mathscr{F}QX \xrightarrow{\mathscr{G}(r_{\mathscr{F}QX})} \mathscr{G}R\mathscr{F}QX$$

The proof that

$$\mathscr{F}Q\mathscr{G}RY \xrightarrow{\mathscr{F}(q_{\mathscr{G}RY})} \mathscr{F}\mathscr{G}RY \xrightarrow{\eta_{RY}} RY \xrightarrow{r_Y^{-1}} Y$$

is an isomorphism follows similarly. One need only check that the diagrams from part (ii) of Proposition 1 commute.

(c) $\Rightarrow$ (a)

This follows formally from part (iii) of Propostion 4. Let X be a cofibrant object of $\mathscr{C}$, Y a cofibrant object of $\mathscr{D}$, and $\varepsilon\colon \mathrm{id}_{\mathrm{Ho}(\mathscr{C})} \to R\mathscr{G} \circ L\mathscr{F}$ the unit of the derived adjunction. Note that by assumption both ε is a natural isomorphism.

Given a morphism $f \in \mathscr{D}(\mathscr{F}X, Y)$ denote its image under the ismorphism of adjunction $\tilde{f} \in \mathscr{C}(X, \mathscr{G}Y)$. We have a commutative diagram

$$\begin{tikzcd}
\mathscr{D}(\mathscr{F}X,Y) \arrow[r, "\sim"] \arrow[d] & \mathscr{C}(X,\mathscr{G}Y) \arrow[d] \\
\mathscr{D}(\mathscr{F}X,Y)/\sim \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathscr{C}(X,\mathscr{G}Y)/\sim \arrow[d, "\sim"] \\
{[L\mathscr{F}X,Y]_{\mathscr{D}}} \arrow[r, "\sim"] & {[X,R\mathscr{G}Y]_{\mathscr{C}}}
\end{tikzcd}$$

and note that f is a weak equivalence if and only if its image in $\mathrm{Ho}(\mathscr{D})$ is an isomorphism. The image of f in $[X, R\mathscr{G}Y]$ is $R\mathscr{G}(f) \circ \varepsilon_X$, which is an isomorphism if and only if $R\mathscr{G}(f)$ is an isomorphism. Noting that $R\mathscr{G}$ reflects isomorphisms because it is fully faithful, we see that $R\mathscr{G}(f)$ is an isomorphism if and only if the image of f in $\mathrm{Ho}(\mathscr{D})$ is an isomorphism. Since $R\mathscr{G}(f) \circ \varepsilon_X$ is the image of $\tilde{f}$ in $\mathrm{Ho}(\mathscr{C})$, it now follows that f is a weak equivalence if and only if $\tilde{f}$ is a weak equivalence. Therefore the adjoint pair $\mathscr{F}, \mathscr{G}$ form a Quillen equivalence. ∎

5. Appendix

Lemma 6. *If X is a cofibrant object of $\mathscr{C}$, then $q_X\colon QX \to X$ is an isomorphism. Dually, if Y is a fibrant object of $\mathscr{C}$, then $r_Y\colon Y \to RY$ is an isomorphism.*

Proof. We have

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & X \\
& QX \arrow[ur, "q_X"] &
\end{tikzcd}$$

with q_X a trivial fibration, which gives a lift

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & QX \arrow[d, "q_X"] \\
X \arrow[r, "\mathrm{id}_X"'] \arrow[ur, dashed, "\exists h"] & X
\end{tikzcd}$$

because X is cofibrant. We note that h is a weak equivalence by 2-out-of-3. By the functorial factorization we get

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\alpha(h)"'] & & QX \\
& X' \arrow[ur, "\beta(h)"'] &
\end{tikzcd}$$

giving a retract

$$\begin{tikzcd}
X \arrow[r, "\alpha(h)"] \arrow[d, "h"] \arrow[rr, bend left, "id_X"] & X' \arrow[r, "q_X \circ \beta(h)"] \arrow[d, "\beta(h)"] & X \arrow[d, "h"] \\
QX \arrow[r, "\mathrm{id}_{QX}"] \arrow[rr, bend right, "\mathrm{id}_{QX}"'] & QX \arrow[r, "\mathrm{id}_{QX}"] & QX
\end{tikzcd}$$

and hence h is a trivial fibration. This now gives a lift

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
QX \arrow[r, "\mathrm{id}_{QX}"'] \arrow[ur, dashed, "\exists h'"] & QX
\end{tikzcd}$$

and so we see

$$q_X = q_X \circ \mathrm{id}_{QX} = q_X \circ h \circ h' = \mathrm{id}_X \circ h' = h'.$$

■