<u>Proposition 1.2.5</u>. Suppose $\mathcal{C}$ is a model category + $f, g : B \longrightarrow X$ are two maps of $\mathcal{C}$.

(i) If $f \overset{\ell}{\sim} g$ + $h : X \longrightarrow Y$, then $hf \overset{\ell}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \longrightarrow B$, then $fh \overset{r}{\sim} gh$.

(ii) If X is fibrant, $f \overset{\ell}{\sim} g$, + $h : A \longrightarrow B$, then $fh \overset{\ell}{\sim} gh$. Dually if B is cofibrant, $f \overset{r}{\sim} g$, + $h : X \longrightarrow Y$, then $hf \overset{r}{\sim} hg$.

(iii) If B is cofibrant, then left homotopy is an equivalence relation on $\mathrm{Mor}_{\mathcal{C}}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation in $\mathrm{Mor}_{\mathcal{C}}(B, X)$.

(iv) If B is cofibrant + $h : X \longrightarrow Y$ is a trivial fibration or a weak equivalence ~~relation~~ of fibrant objects, then h induces an isomorphism $\mathrm{Mor}_{\mathcal{C}}(B, X)/\overset{\ell}{\sim} \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{C}}(B, Y)/\overset{\ell}{\sim}$. Dually, if X is fibrant and $h : A \longrightarrow B$ is a trivial fibration or a weak equivalence of cofibrant objects, then h induces an isomorphism $\mathrm{Mor}_{\mathcal{C}}(B, X)/\overset{r}{\sim} \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{C}}(A, X)/\overset{r}{\sim}$.

(v) If B is cofibrant, then $f \overset{\ell}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, there is a right homotopy $K : B \longrightarrow X'$ from f to g. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{\ell}{\sim} g$, + there is a left homotopy from f to g using any cylinder object for B.

Pf (v) Assume B is cofibrant and $f \overset{\ell}{\sim} g : B \longrightarrow X$. Since $f \overset{\ell}{\sim} g$, there exists some cylinder object B' + $H : B' \longrightarrow X$ s.t. $H i_0 = f$ and $H i_1 = g$.

$$\begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \\ & B' \arrow[u, "s"] \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

Note $i_0 \amalg i_1$ is a cofibration + s is a w.e.

Let $X \xrightarrow{r} X' \xrightarrow{(p_0, p_1)} X \times X$ be any path object for X. We want to find $K : B \to X'$ a right homotopy from f to g. Have the following:

$$\begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[rr, "\Delta"] \arrow[dr, "r"'] & & X \times X \\ & B' \arrow[u, "s"] \arrow[ur, dashed, "H"'] & & X' \arrow[ur, "(p_0, p_1)"'] & \end{tikzcd}$$

r = w.e. and (p_0, p_1) = fibration.

②

We have the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "rf"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "(p_0, p_1)"] \\ B' \arrow[r, "(fs, H)"'] & X \times X \end{tikzcd}$$

to find our $K: B \to X'$, we will first find a lift $J: B' \to X'$. We need to show i_0 is a trivial cofibration.

Let ϕ Denote the initial object in $\mathcal{C}$. Then $\phi \to B$ is a cofibration + we have the following pushout:

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\varphi"] \arrow[ddr, bend left, "i_0"] & \\ B \arrow[r] \arrow[drr, bend right, "i_1"'] & B \amalg B \arrow[dr, dashed, "i_0 \amalg i_1"] & \\ & & B' \end{tikzcd}$$

where $i_0 = (i_0 \amalg i_1) \circ \varphi$.

φ is a cofibration b/c cofibrations are closed under pushouts (corollary 1.1.11) and $(i_0 \amalg i_1)$ is a cofibration by assumption. Since cofibrations form a subcategory, i_0 is a cofibration.

From the fact that B' is a cylinder object, we have the commutative triangle:

$$\begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & B \\ & B' \arrow[u, "s"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

Restricting ∇ to one copy of B, which is 1_B, gives:

$$\begin{tikzcd} B \arrow[r, "1_B"] \arrow[dr, "i_0"'] & B \\ & B' \arrow[u, "s"'] \end{tikzcd}$$

Thus $s \circ i_0 = 1_B$. Since s is a w.e. and 1_B is a w.e., i_0 is a w.e. by 2-3.

$\therefore$ i_0 is a trivial cofibration.

3

We can conclude that $\exists\, J : B' \longrightarrow X'$, a lift in the diagram

$$\begin{tikzcd}
B \arrow[r, "rf"] \arrow[d, "i_0"'] & X' \arrow[d, "(p_0, p_1)"] \\
B' \arrow[r, "(fs, H)"'] \arrow[ur, dashed, "J"] & X \times X
\end{tikzcd}$$

Then we have:

$$\begin{tikzcd}
B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[rr, "\Delta"] \arrow[dr, "r"] & & X \times X \\
& B' \arrow[u, "s"] \arrow[ur, "H"] \arrow[rr, dashed, "J"'] & & X' \arrow[ur, "(p_0, p_1)"'] &
\end{tikzcd}$$

Let $K = J \circ i_1 : B \longrightarrow X'$. Then $p_0 \circ K = p_0 \circ J \circ i_0 = f$ and $p_1 \circ K = p_1 \circ J \circ i_1 = g$. $\therefore$ K is a right homotopy, as desired. $\blacksquare$

We have 2 immediate corollaries:

<u>Corollary (1.2.6)</u>. Suppose $\mathcal{C}$ is a model category, $B \in \mathcal{C}$ is cofibrant, and $X \in \mathcal{C}$ is fibrant. Then the left homotopy relation + right homotopy relation coincide + are equivalence relations on $\mathrm{Mor}_{\mathcal{C}}(B, X)$. If $f \sim g : B \longrightarrow X$, then there is a left homotopy $H : B' \longrightarrow X$ from f to g using any cylinder object B' of B. Dually, there is a right homotopy $K : B \longrightarrow X'$ using any path object X' for B.

Corollary (1.2.7). The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation + is compatible w/ composition. Hence the category $\mathcal{C}_{cf}/\sim$ exists.

Proposition (1.2.8). Suppose $\mathcal{C}$ is a model category. Then a map of $\mathcal{C}_{cf}$ is a weak equivalence if + only if it is a homotopy equivalen

Pf. ($\Rightarrow$) Assume first that $f: A \longrightarrow B$ is a weak equivalence in $\mathcal{C}_{cf}$. Since A, B are fibrant + any arbitrary $X \in \mathcal{C}_{cf}$ is cofibrant, f induces an isomorphism $Mor_{\mathcal{C}_{cf}}(X,A) \xrightarrow{\sim} Mor_{\mathcal{C}_{cf}}(X,B)$. By Ken Brown's lemma (applied to $\mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$), we have an isomorphism $Mor_{\mathcal{C}_{cf}/\sim}(X,A) \xrightarrow{\sim} Mor_{\mathcal{C}_{cf}/\sim}(X,B)$. Take $X = B$.

then $\exists g: B \longrightarrow A$ unique up to homotopy in $\mathcal{C}_{cf}/\sim$ s.t. $g \mapsto 1_B$ under this isomorphism and $fg \sim 1_B$. By Prop 1.2.5(ii), $fgf \sim f$. Similar reasoning w/ $X = A \Rightarrow gf \sim 1_A$. $\therefore$ f is a homotopy equivalence.

($\Leftarrow$) Assume $f: A \longrightarrow B$ is a homotopy equivalence in $\mathcal{C}_{cf}$. Since $\mathcal{C}$ is a model cat, we can factor f as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

where g is a trivial cofibration + p is a fibration. We claim $C \in \mathcal{C}_{cf}$.

$$\begin{tikzcd} & \phi \arrow[dl, "\alpha"'] \arrow[dd, "\gamma"] \arrow[dr] & \\ A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

Since $\gamma = g\alpha$, g = cofibration, + α = cofibration by assumption since A is cofibrant, γ is a cofibration. $\therefore$ C is cofibrant.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[dr, "\beta'"] & \\ & C \arrow[u, "p"'] \arrow[r, bend right, "\gamma'"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & * \end{tikzcd}$$

Since γ' is unique, $\gamma' = \beta' p$. Since p is a fibration & β' is a fibration, γ' is a fibration. Thus C is fibrant.

We conclude that $C \in \mathcal{C}_{cf}$. By the forward direction of this proposition, g is a homotopy equivalence. To show f is a w.e., it now suffices to show p is a w.e. by 2-3. Let $f': B \longrightarrow A$ be a homotopy inverse for f. Then $ff' \overset{l}{\sim} 1_B : B \longrightarrow B$. Let $H: B' \longrightarrow B$ be the left homotopy $ff' \overset{l}{\sim} 1_B$:

$$\begin{tikzcd} B \amalg B \arrow[r, "\nabla"] \arrow[dr, bend right, "i_0 \amalg i_1"'] & B \arrow[r, shift left, "ff'"] \arrow[r, shift right, "1_B"'] & B \\ & B' \arrow[u, "s"] \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

We have the following commutative square:

$$\begin{tikzcd} B \arrow[r, "gf'"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "p"] \\ B' \arrow[r, "H"'] & B \end{tikzcd}$$

we would like to show i_0 is a trivial cofibration to find a lift.

(Literally same as in proof of Prop 1.2.5)

Thus $\exists H': B' \longrightarrow C$, a lift in the diagram:

$$\begin{tikzcd} B \arrow[r, "gf'"] \arrow[d, "i_0"'] & C \arrow[d, "p"] \\ B' \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & B \end{tikzcd}$$

⑥

Let $q := H' i_1 : B \longrightarrow C$. By the commutivity of the lift, $H' i_0 = g f'$.

Thus H' is a left homotopy $gf' \overset{\ell}{\sim} q$

$$\begin{tikzcd}
B \amalg B \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & B \arrow[r, shift left, "gf'"] \arrow[r, shift right, "q"'] & C \\
& B' \arrow[u, "s"] \arrow[ur, dashed, "H'"'] &
\end{tikzcd}$$

Note that since g is a w.e., g has a homotopy inverse $g' : C \longrightarrow A$. Then $p \sim p 1_C \sim p(gg') = fg'$. Hence

$qp \sim q(fg') \sim (gf')(fg') \sim 1_C$. Therefore $qp \overset{\ell}{\sim} 1_C$.

Let $K : C' \longrightarrow C$ be a left homotopy between qp and 1_C, C' some cylinder object of C.

$$\begin{tikzcd}
C \amalg C \arrow[r, "\nabla"] \arrow[dr, "j_0 \amalg j_1"'] & C \arrow[r, shift left, "qp"] \arrow[r, shift right, "1_C"'] & C \\
& C' \arrow[u, "t"] \arrow[ur, dashed, "K"'] &
\end{tikzcd}$$

Note $K j_0 = 1_C$ and $K j_1 = qp$. By the same reasoning as why i_0 is a w.e., j_0 is a w.e. Also 1_C is a w.e. By 2-3, K is a w.e.

Thus $qp = K j_1$ is a w.e. by 2-3.

We have a retract:

$$\begin{tikzcd}
C \arrow[r, "1_C"] \arrow[d, "p"'] & C \arrow[r, "1_C"] \arrow[d, "qp"] & C \arrow[d, "p"] \\
B \arrow[r, "q"] & C \arrow[r, "p"] & B
\end{tikzcd}$$

By the retract assumption of a model structure, p is a w.e. Thus $f = pg$ is a w.e.

$\square$

Corollary (1.2.9). Let $\mathcal{C}$ be a model category. Let $\gamma : \mathcal{C}_{cf} \longrightarrow Ho\,\mathcal{C}_{cf}$ and $\delta : \mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$ be the canonical functors. There exists a unique $j : \mathcal{C}_{cf}/\sim \xrightarrow{\sim} Ho\,\mathcal{C}_{cf}$ s.t. $j\delta = \gamma$. j is the identity on objects.

Pf. We will show that $\mathcal{C}_{cf}/\sim$ satisfies the same universal property as $Ho\,\mathcal{C}_{cf}$. Let $\delta : \mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$ be defined as the identity on objects + homotopy equivalences map to isomorphisms. Note that by Prop 1.2.8, w.e. in $\mathcal{C}_{cf}$ also map to isomorphisms in $\mathcal{C}_{cf}/\sim$.

Let $F : \mathcal{C}_{cf} \longrightarrow \mathcal{D}$ be any functor that maps w.e. to isomorphisms. Let $A \in \mathcal{C}_{cf}$ and let A' be a cylinder object for A:

$$\begin{tikzcd} A \amalg A \arrow[r, "\nabla"] \arrow[dr, "i_0 \amalg i_1"'] & A \\ & A' \arrow[u, "s"'] \end{tikzcd}$$

restricting ∇ to any one copy of A, $s i_0 = s i_1 = 1_A$. Note $s = $ w.e. $\Rightarrow$ $F(s)$ an iso. Thus

$$F(s)\,F(i_0) = F(s i_0) = F(s i_1) = F(s)\,F(i_1), \text{ so}$$

$$F(i_0) = F(i_1).$$

Thus if $H : A' \longrightarrow B$ is a left homotopy between two $f, g : A \longrightarrow B$ in $\mathcal{C}_{cf}$, then $H i_0 = f$ + $H i_1 = g$ implies $F(f) = F(H i_0) = F(H i_1) = F(g)$. $\therefore F$ identifies left + right homotopic maps. Define $G : \mathcal{C}_{cf}/\sim \longrightarrow \mathcal{D}$ by $c \mapsto c$ and $f \mapsto F(f)$. Then $G\delta = F$. By Lemma 1.2.2, $\exists!$ $Ho\,\mathcal{C}_{cf} \xrightarrow{\sim} \mathcal{C}_{cf}/\sim$ that is an isomorphism. Let

$j : \mathcal{C}_{cf}/\sim \longrightarrow Ho\,\mathcal{C}_{cf}$ be its inverse.

$\square$