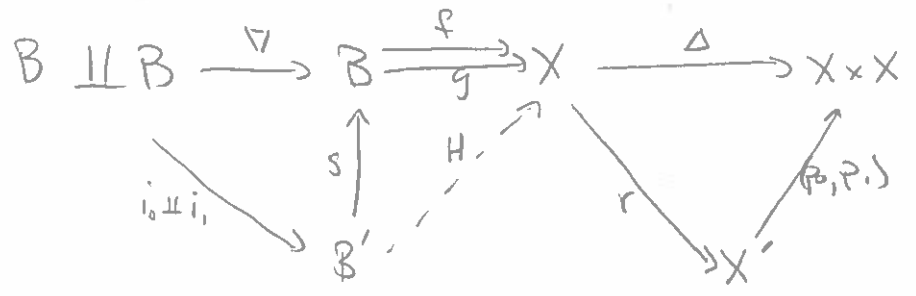


Proposition 1.2.5. Suppose \mathcal{C} is a model category + $f, g: B \rightarrow X$ are two maps of \mathcal{C} .

- (i) If $f \stackrel{L}{\sim} g$ + $h: X \rightarrow Y$, then $hf \stackrel{L}{\sim} hg$. Dually, if $f \stackrel{R}{\sim} g$ and $h: A \rightarrow B$, then $fh \stackrel{R}{\sim} gh$.
- (ii) If X is fibrant, $f \stackrel{L}{\sim} g$, + $h: A \rightarrow B$, then $fh \stackrel{L}{\sim} gh$. Dually if B is cofibrant, $f \stackrel{R}{\sim} g$, + $h: X \rightarrow Y$, then $hf \stackrel{R}{\sim} hg$.
- (iii) If B is cofibrant, then left homotopy is an equivalence relation on $\text{Mor}_e(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation in $\text{Mor}_e(B, X)$.
- (iv) If B is cofibrant + $h: X \rightarrow Y$ is a trivial fibration or a weak equivalence ~~of~~ of fibrant objects, then h induces an isomorphism $\text{Mor}_e(B, X) / \sim \xrightarrow{\cong} \text{Mor}_e(B, Y) / \sim$. Dually, if X is fibrant and $h: A \rightarrow B$ is a trivial fibration or a weak equivalence of cofibrant objects, then h induces an isomorphism $\text{Mor}_e(B, X) / \sim \xrightarrow{\cong} \text{Mor}_e(A, X) / \sim$.
- (v) If B is cofibrant, then $f \stackrel{L}{\sim} g$ implies $f \stackrel{R}{\sim} g$. Furthermore, if X' is any path object for X , there is a right homotopy $K: B \rightarrow X'$ from f to g . Dually, if X is fibrant, then $f \stackrel{R}{\sim} g$ implies $f \stackrel{L}{\sim} g$, + there is a left homotopy from f to g using any cylinder object for B .

~~Proof~~ (v) Assume B is cofibrant and $f \stackrel{L}{\sim} g: B \rightarrow X$. Since $f \stackrel{L}{\sim} g$, there exists some cylinder object $B' \rightrightarrows B$ s.t. $H \circ i_0 = f$ and $H \circ i_1 = g$. Note $i_0 \sqcup i_1$ is a cofibration + δ is a w.e.

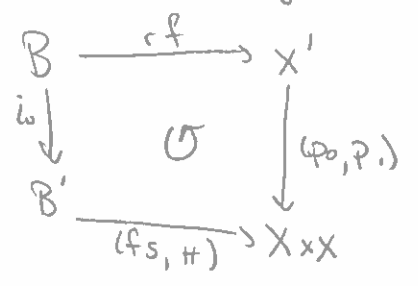
Let $X \rightrightarrows X' \xrightarrow{(p_0, p_1)} X \times X$ be any path object for X . We want to find $K: B \rightarrow X'$ a right homotopy from f to g . Have the following:



$r = \text{w.e.}$ and $(p_0, p_1) = \text{fibration}$.

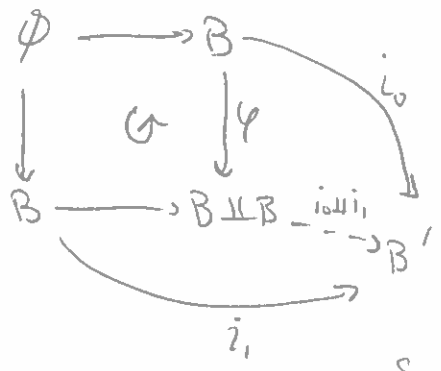
2

We have the following commutative diagram:



to find our $K: B \rightarrow X'$, we will first find a lift $J: B' \rightarrow X'$. we need to show i_0 is a trivial cofibration.

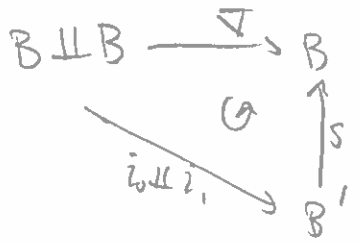
Let ϕ denote the initial object in \mathcal{C} . Then $\phi \rightarrow B$ is a cofibration & we have the following pushout:



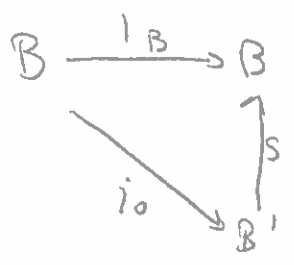
where $i_0 = (i_0 \amalg i_1) \circ \varphi$. φ is a cofibration b/c cofibrations are closed under pushouts (corollary 1.1.11) and $(i_0 \amalg i_1)$ is a cofibration by assumption. Since cofibrations form a subcategory,

i_0 is a cofibration.

From the fact that B' is a cylinder object, we have the commutative triangle:



Restricting ∇ to one copy of B , which is l_B , gives:

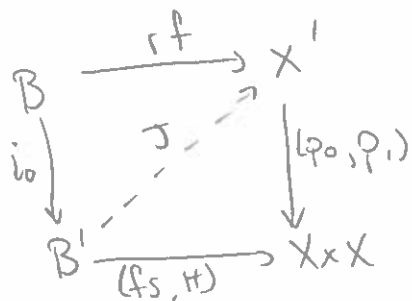


Thus $s \circ i_0 = l_B$. Since s is a w.e. and l_B is a w.e., i_0 is a w.e. by 2-3.

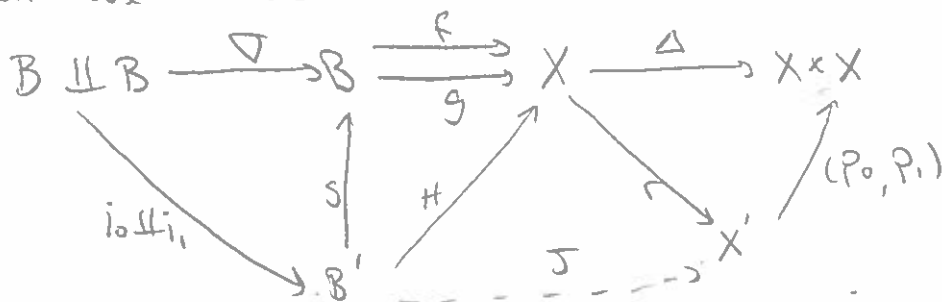
$\therefore i_0$ is a trivial cofibration.

(3)

We can conclude that $\exists J: B' \rightarrow X'$, a lift in the diagram



Then we have:



Let $K = J \circ i_1: B \rightarrow X'$. Then $p_0 \circ K = p_0 \circ J \circ i_0 = f$ and $p_1 \circ K = p_1 \circ J \circ i_1 = g$. $\therefore K$ is a right homotopy, as desired. \square

We have an immediate corollary:

Corollary (1.2.6). Suppose \mathcal{C} is a model category, $B \in \mathcal{C}$ is cofibrant, and $X \in \mathcal{C}$ is fibrant. Then the left homotopy relations + right homotopy relations coincide + are equivalence relations on $\text{Map}_{\mathcal{C}}(B, X)$. If $f \sim g: B \rightarrow X$, then there is a left homotopy $H: B' \rightarrow X$ from f to g using any cylinder object B' of B . Dually, there is a right homotopy $K: B \rightarrow X'$ using any path object X' for B .

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Corollary (1.2.7). The homotopy relation on the morphisms of \mathcal{C}_{cf} is an equivalence relation + is compatible w/ composition. Hence the category \mathcal{C}_{cf}/\sim exists.

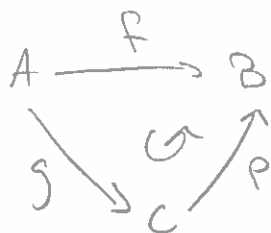
Proposition (1.2.8). Suppose \mathcal{C} is a model category. Then a map of \mathcal{C}_{cf} is a weak equivalence if + only if it is a homotopy equivalence.

\Rightarrow Assume first that $f: A \rightarrow B$ is a weak equivalence in \mathcal{C}_{cf} . Since A, B are fibrant + any arbitrary $X \in \mathcal{C}_{cf}$ is cofibrant, f induces an isomorphism $\text{Mor}_{\mathcal{C}_{cf}}(X, A) \xrightarrow{\sim} \text{Mor}_{\mathcal{C}_{cf}}(X, B)$. By Ken Brown's lemma (applied to $\mathcal{C}_{cf} \rightarrow \mathcal{C}_{cf}/\sim$), we have an isomorphism $\text{Mor}_{\mathcal{C}_{cf}/\sim}(X, A) \xrightarrow{\sim} \text{Mor}_{\mathcal{C}_{cf}/\sim}(X, B)$. Take $X=B$.

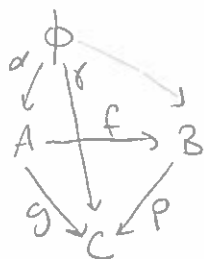
then $\exists g: B \rightarrow A$ unique up to homotopy in \mathcal{C}_{cf}/\sim s.t. $g \mapsto 1_B$ under this isomorphism and $fg \sim 1_B$. By Prop 1.2.5(ii), $fgf \sim f$. Similar reasoning w/ $X=A \Rightarrow gf \sim 1_A \therefore f$ is a homotopy equivalence.

\Leftarrow Assume $f: A \rightarrow B$ is a homotopy equivalence in \mathcal{C}_{cf} .

Since \mathcal{C} is a model cat, we can factor f as

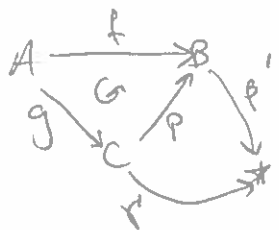


where g is a trivial cofibration + p is a fibration. We claim $C \in \mathcal{C}_{cf}$.



Since $\gamma = g\alpha$, $g = \text{cofibration}$, + $\alpha = \text{cofibration}$ by assumption since A is cofibrant, γ is a cofibration. $\therefore C$ is cofibrant.

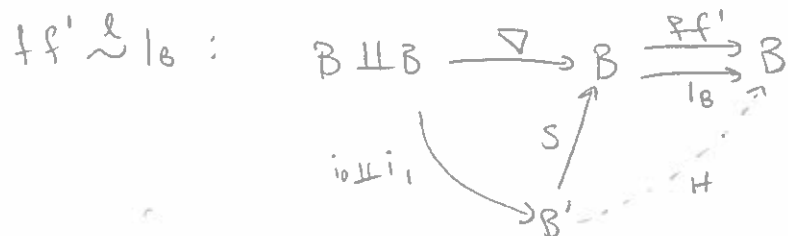
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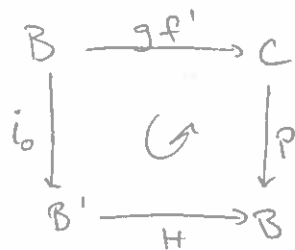
Since r' is unique, $r' = \beta' p$. Since p is a fibration & β' is a fibration, r' is a fibration. Thus C is fibrant.

We conclude that $C \in \mathcal{C}_{cf}$. By the forward direction of this proposition, g is a homotopy equivalence. To show f is a w.e., it now suffices to show p is a w.e. by 2-3.

Let $f': B \rightarrow A$ be a homotopy inverse for f . Then $f f' \sim 1_B: B \rightarrow B$. Let $H: B' \rightarrow B$ be the left homotopy



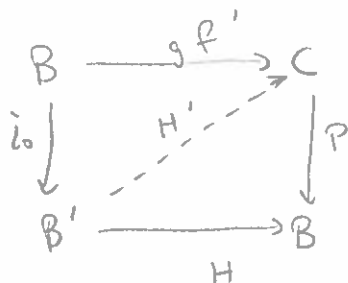
We have the following commutative square:



we would like to show i_0 is a trivial cofibration to find a lift.

(Literally same as in proof of Prop 1.2.5)

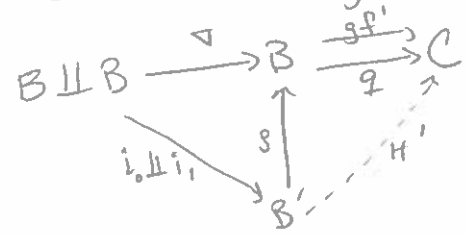
Thus $\exists H': B' \rightarrow C$, a lift in the diagram:



⑥

Let $q := H' i_1 : B \rightarrow C$. By the commutativity of the left, $H' i_0 = q f'$.

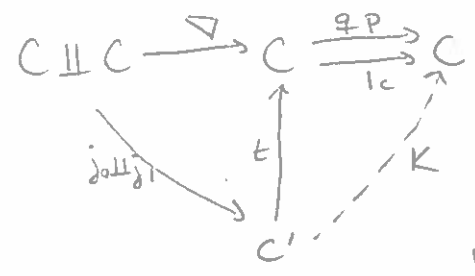
Thus H' is a left homotopy $q f' \stackrel{H'}{\sim} q$



Note that since q is a w.e., q has a homotopy inverse $g' : C \rightarrow A$. Then $p \sim p l_c \sim p (q g') = f g'$. Hence

$$q p \sim q (f g') \sim (q f') (g') \sim l_c. \text{ Therefore } q p \sim l_c.$$

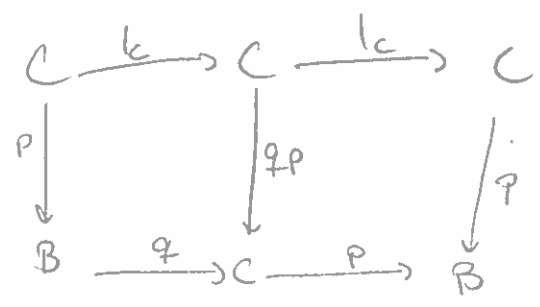
Let $K : C' \rightarrow C$ be a left homotopy between $q p$ and l_c , C' some cylinder object of C .



Note $K j_0 = l_c$ and $K j_1 = q p$. By the same reasoning as why i_0 is a w.e., j_0 is a w.e. Also l_c is a w.e. By 2-3, K is a w.e.

Thus $q p = K j_1$ is a w.e. by 2-3.

We have a retract:



By the retract assumption of a model structure, p is a w.e. Thus $t = p q$ is a w.e.

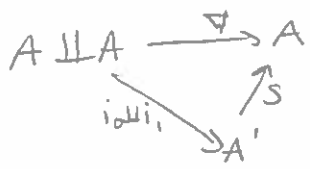
□

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Corollary (1.2.9). Let \mathcal{C} be a model category. Let $\gamma: \mathcal{C}_{cf} \rightarrow \text{Ho } \mathcal{C}_{cf}$ and $\delta: \mathcal{C}_{cf} \rightarrow \mathcal{C}_{cf}/\sim$ be the canonical functors. There exists a unique $j: \mathcal{C}_{cf}/\sim \xrightarrow{\sim} \text{Ho } \mathcal{C}_{cf}$ s.t. $j\delta = \gamma$. j is the identity on objects.

pf. We will show that \mathcal{C}_{cf}/\sim satisfies the same universal property as $\text{Ho } \mathcal{C}_{cf}$. Let $\delta: \mathcal{C}_{cf} \rightarrow \mathcal{C}_{cf}/\sim$ be defined as the identity on objects + homotopy equivalences map to isomorphisms. Note that by Prop 1.2.8, w.e. in \mathcal{C}_{cf} also map to isomorphisms in \mathcal{C}_{cf}/\sim .

Let $F: \mathcal{C}_{cf} \rightarrow \mathcal{D}$ be any functor that maps w.e. to isomorphisms. Let $A \in \mathcal{C}_{cf}$ and let A' be a cylinder object for A :



restricting ∇ to any one copy of A , $s i_0 = s i_1 = 1_A$. Note $s = \text{w.e.} \Rightarrow F(s)$ an iso. Thus

$$F(s) F(i_0) = F(s i_0) = F(s i_1) = F(s) F(i_1), \text{ so } F(i_0) = F(i_1).$$

Thus if $H: A' \rightarrow B$ is a left homotopy between two $f, g: A \rightarrow B$ in \mathcal{C}_{cf} , then $H i_0 = f + H i_1 = g$ implies $F(f) = F(H i_0) = F(H i_1) = F(g)$. $\therefore F$ identifies left + right homotopic maps. Define $G: \mathcal{C}_{cf}/\sim \rightarrow \mathcal{D}$ by $c \mapsto c$ and $f \mapsto F(f)$. Then $G\delta = F$. By Lemma 1.2.2, $\exists!$ $\text{Ho } \mathcal{C}_{cf} \xrightarrow{\sim} \mathcal{C}_{cf}/\sim$ that is an isomorphism. Let $j: \mathcal{C}_{cf}/\sim \rightarrow \text{Ho } \mathcal{C}_{cf}$ be its inverse. □