ON THE VANISHING OF FOURIER COEFFICIENTS OF CERTAIN GENUS ZERO NEWFORMS

MATTHEW BOYLAN, SHARON ANNE GARTHWAITE, AND JOHN WEBB

Abstract. Given a classical modular form $f(z)$, a basic question is whether any of its Fourier coefficients vanish. This question remains open for certain modular forms. For example, let $\Delta(z) = \sum \tau(n)q^n \in S_{12}(\Gamma_0(1))$. A well-known conjecture of Lehmer asserts that $\tau(n) \neq 0$ for all n. In recent work, Ono constructed a family of polynomials $A_n(x) \in \mathbb{Q}[x]$ with the property that $\tau(n)$ vanishes if and only if $A_n(0)$ and $A_n(1728)$ do. In this paper, we establish a similar criterion for the vanishing of coefficients of certain newforms on genus zero groups of prime level.

1. Introduction

Let $z \in \mathbb{H}$, the complex upper-half plane, let $q := e^{2\pi i z}$, and let N and k be positive integers. A classical modular form $f(z)$ of weight k on the congruence subgroup $\Gamma_0(N)$ has a q-expansion $f(z) = \sum a_f(n)q^n$. Of some interest are questions as to when, if ever, the coefficients $a_f(n)$ vanish.

We completely understand the vanishing behavior of coefficients of certain types of modular forms. For integers $j \geq 0$ we denote the jth Bernoulli number by B_j. For even integers $k \geq 2$, we define the Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n. \tag{1.1}$$

Further, we agree that $E_0(z) := 1$. When $k \geq 4$, $E_k(z)$ is a modular form, none of whose coefficients vanish. At the other extreme, one calls a modular form lacunary if a density one subset of its coefficients vanish. Serre [17] proved that an integer weight modular form is lacunary if and only if it is a linear combination of modular forms with complex multiplication; he later used this criterion [18] to prove that $\prod(1-q^n)^r$ with even $r \geq 2$ is lacunary if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$.

On the other hand, for some modular forms $f(z)$, it is not known whether any coefficients $a_f(n)$ vanish. Lehmer's Conjecture furnishes the most famous example of questions of this type. Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{1.2}$$

Date: August 25, 2010.

1991 *Mathematics Subject Classification.* 11F03, 11F11.

The first author thanks the National Science Foundation for its support through grant DMS-0901068.

denote the Dedekind eta-function. We define the Delta-function by

$$\Delta(z) := \eta(z)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

It is a cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. Lehmer's Conjecture asserts that $\tau(n) \neq 0$ for all n.

In recent work, Ono [13, 14] constructed a family of polynomials $A_n(x) \in \mathbb{Q}[x]$ for all $n \geq 1$ with the property that $\tau(n)$ vanishes if and only if $A_n(0)$ and $A_n(1728)$ do. In this paper, we establish a similar criterion for the vanishing of coefficients of certain newforms on genus zero groups of prime level. Moreover, we highlight deeper connections our construction has to the theory of harmonic weak Maass forms. To state our results we first require some notation and definitions.

In this paper, we consider $N \in S := \{1, 2, 3, 5, 7, 13\}$; for these N, the modular curve $X_0(N)$ has genus zero. In addition, for $N \neq 1$ in S, the congruence subgroup $\Gamma_0(N)$ has two inequivalent cusps, represented by zero and infinity. We denote the complex vector space of weakly holomorphic modular forms of weight k on $\Gamma_0(N)$ by $M_k^!(\Gamma_0(N)$. Forms in $M_k^!(\Gamma_0(N))$ have poles, if any, supported at cusps. We denote by $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ the subspaces of holomorphic modular forms and cusp forms, respectively.

For $N \in S$ and positive integers k, we require certain Eisenstein series in $M_k(\Gamma_0(N))$. If χ is a Dirichlet character modulo N and j and n are integers with $j \geq 0$, we define

$$\sigma_{\chi,j}(n) := \sum_{d|n} \chi(d) d^j.$$

We denote by χ_N^{triv} the trivial character modulo N, and we set $\sigma_j(n) := \sigma_{\chi_1^{\mathrm{triv}},j}(n)$. For $N \neq 1$ in S, we define

$$E_{N,2}^*(z) := \frac{NE_2(Nz) - E_2(z)}{N-1} = 1 + \frac{24}{N-1} \sum_{n=1}^{\infty} \sigma_{\chi_N^{\mathrm{triv}},1}(n) q^n \in M_2(\Gamma_0(N)). \tag{1.3}$$

For $N \in S$ and weights $k > 2$, we define

$$E_{N,k,\infty}(z) = 1 + \cdots := \begin{cases} E_k(z) & N = 1, \\ \frac{N^k E_k(Nz) - E_k(z)}{N^k - 1} & N \in \{2, 3, 5, 7, 13\}. \end{cases} \tag{1.4}$$

Note that for $N \neq 1$ in S, the form $E_{N,k,\infty}(z)$ does not vanish at infinity, but vanishes at the cusp zero. Moreover, for such N, it follows from (1.1) and (1.4) that

$$E_{N,k,\infty}(z) = 1 - \frac{2k}{B_k(N^k - 1)} \sum_{n=1}^{\infty} \left(N^k \sigma_{k-1}\left(\frac{n}{N}\right) - \sigma_{k-1}(n) \right) q^n. \tag{1.5}$$

For $N \neq 1$ in S, we also require

$$E_{N,k,0}(z) = q + \cdots := \frac{B_k}{2k}(E_k(Nz) - E_k(z)) \in M_k(\Gamma_0(N)). \tag{1.6}$$

Note that $E_{N,k,0}(z)$ does not vanish at the cusp zero, but vanishes at infinity. We have

$$E_{N,k,0}(z) = \sum_{n=1}^{\infty} \left(\sigma_{k-1}(n) - \sigma_{k-1}\left(\frac{n}{N}\right) \right) q^n.$$

Now, let $N \in S$ and let $k \in \mathbb{Z}$. If $S_k(\Gamma_0(N))$ is one-dimensional, we denote its normalized generator by

$$f_{N,k}(z) = \sum_{n=1}^{\infty} a_{N,k}(n) q^n = q + \cdots . \tag{1.7}$$

In the following table, we list all one-dimensional spaces $S_k(\Gamma_0(N))$ and the corresponding $f_{N,k}(z)$ in terms of (1.1), (1.2), (1.3), (1.6) and (1.9) (see below for the definition of (1.9)).

N	k	$f_{N,k}(z)$
1	$12, 16, 18, 20, 22, 26$	$\eta(z)^{24} E_{k-12}(z)$
2	8	$\eta(z)^8 \eta(2z)^8$
2	10	$f_{2,8}(z) E^*_{2,2}(z)$
3	6	$\eta(z)^6 \eta(3z)^6$
3	8	$f_{3,6}(z) E^*_{3,2}(z)$
5	4	$\eta(z)^4 \eta(5z)^4$
5	6	$f_{5,4}(z) E^*_{5,2}(z)$
7	4	$\frac{10}{51}\left(\phi_7(z) E_{7,4,0}(z) - E_{7,4,\infty}(z)\right)$

In [16], Ono presents a table with similar information for all even weight eta-product newforms, a list that contains our $f_{1,12}(z)$, $f_{2,8}(z)$, $f_{3,6}(z)$, and $f_{5,4}(z)$. We have checked for vanishing among $a_{N,k}(n)$ for $n < 10{,}000$, and the only form to have vanishing coefficients for such n is $f_{5,4}(z)$. One can check for $n = 2^r m$ with $r \equiv 3 \pmod 4$ and m odd that $a_{5,4}(n) = 0$.

Moreover, congruences for the coefficients of $f_{N,k}(z)$ restrict the possible n for which $a_{N,k}(n)$ could vanish. For example, let $N = 2$ or $N = 3$ and suppose that ℓ is prime with $\ell \mid B_k(N^k - 1)$ but $\ell \nmid 2k$. One can show for all primes $p \neq N$ that

$$a_{N,k}(p) \equiv 1 + p^{k-1} \pmod{\ell}. \tag{1.8}$$

Hence, if $a_{N,k}(p) = 0$ then we must have $p^{k-1} \equiv -1 \pmod{\ell}$, which implies for $(N, k, \ell) \in \{(2, 8, 17), (3, 6, 13), (3, 8, 41)\}$ that $p \equiv -1 \pmod{\ell}$. Similar congruence restrictions hold for the coefficients of the other forms in the table.

We now turn our attention to defining $\phi_N(z) \in M_0^!(\Gamma_0(N)) \cap \mathbb{Z}((q))$, a suitable generator for the field of rational functions on $X_0(N)$. For this purpose, take

$$\phi_N(z) := \begin{cases} j(z) = E_4(z)^3 \Delta(z)^{-1} \text{ if } N = 1, \\ \left(\frac{\eta(z)}{\eta(Nz)}\right)^{\frac{24}{N-1}} \quad \text{if } N \neq 1 \text{ in } S. \end{cases} \tag{1.9}$$

Next, for $N \in S$ and $n \geq 0$ we define a sequence of forms $j_{N,n}(z) \in M_0^!(\Gamma_0(N))$ as follows. Let $f(z)$ be a meromorphic function on $\mathbb{H}$, let $\tau \in \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, and denote by $\nu_\tau(f(z))$ the order of vanishing at τ. Set $j_{N,0}(z) := 1$, and for $n \geq 1$, define $j_{N,n}(z)$ to be a form in $M_0^!(\Gamma_0(N))$ for which

$$\begin{aligned} &\nu_0(j_{N,n}(z)) \geq 0, \\ &j_{N,n}(z) = q^{-n} + c_{N,n} + O(q). \end{aligned} \tag{1.10}$$

Observe that $j_{N,n}(z)$ is the unique form in $M_0^!(\Gamma_0(N))$ with these properties. To see this, note that if $f(z)$ is a second form with these properties, then the difference $j_{N,n}(z) - f(z)$ is a holomorphic modular form of weight zero which vanishes at infinity. Hence, the difference is constant; in particular, it must be zero. Further, note that $j_{N,1}(z) = \phi_N(z)$.

We now define a related sequence $J_{N,n}(z) \in M_0^!(\Gamma_0(N))$. For all $N \in S$ define $J_{N,0}(z) := 1$, and for all $n \geq 1$, set

$$J_{N,n}(z) := j_{N,n}(z) - c_{N,n} = q^{-n} + O(q). \tag{1.11}$$

We remark that for positive integers n with $N \nmid n$, we have

$$J_{N,n}(z) = J_{N,1}(z) \mid nT_{N,0}(n),$$

where $T_{N,0}(n)$ is the usual Hecke operator with level N, weight 0, and index n. Now, since $j_{N,1}(z)$ generates the space $M_0^!(\Gamma_0(N))$, and since $J_{N,n}(z) \in M_0^!(\Gamma_0(N))$ satisfies $\nu_0(J_{N,n}(z)) \geq 0$ and $\nu_\infty(J_{N,n}(z)) = -n$, it follows that $J_{N,n}(z)$ is expressible as a monic polynomial in $j_{N,1}(z)$ with integer coefficients. Hence, for all integers $N \in S$ and $m \geq 0$, we define monic polynomials $p_{N,m}(x) \in \mathbb{Z}[x]$ as follows. Set $p_{N,0}(x) := 1$ and for $m \geq 1$, define $p_{N,m}(x)$ by

$$p_{N,m}(j_{N,1}(z)) = J_{N,m}(z). \tag{1.12}$$

We will study the generating function for these polynomials, given by

$$H_{N,x}(z) := \sum_{n=0}^{\infty} p_{N,n}(x) q^n; \tag{1.13}$$

the polynomials $p_{1,m}(x)$ are usually called Faber polynomials.

Finally, for pairs (N,k) with $S_k(\Gamma_0(N))$ one-dimensional, choose $G = 1 + \cdots = \sum b_G(n) q^n$ in $M_{k-2}(\Gamma_0(N))$. For $m \geq 0$, and with $p_{N,m}(x)$ as in (1.12), define polynomials $A_{N,k,G,m}(x)$ by

$$\sum_{m=0}^{\infty} A_{N,k,G,m}(x) q^m := H_{N,x}(z) \cdot G(z) - E_{N,k,\infty}(z). \tag{1.14}$$

When $N = 1$, the mth q-expansion coefficient of $E_{N,k,\infty}(z)$, which we denote by $\beta_{N,k}(m)$, is given by (1.1). For $N \neq 1$ in S, the mth coefficient of $E_{N,k,\infty}(z)$ is given by (1.5). It follows from (1.14) that

$$A_{N,k,G,m}(x) = \sum_{n=0}^{m} b_G(n) p_{N,m-n}(x) - \beta_{N,k}(m). \tag{1.15}$$

We now use the notation and definitions above to concisely state the known results on vanishing criteria for newform coefficients in terms of the polynomials in (1.14).

Theorem 1.1. *Assume the notation above. Let $N = 1$, let m be a positive integer, let $k \in \{12, 16, 18, 20, 22, 26\}$, and fix $G(z) = E_{k-2}(z)$.*

(1) *If $k \equiv 0, 4 \pmod 6$, then we have $a_{1,k}(m) = 0$ if and only if $A_{1,k,G,m}(0) = 0$.*
(2) *If $k \equiv 0 \pmod 4$, then we have $a_{1,k}(m) = 0$ if and only if $A_{1,k,G,m}(1728) = 0$.*

Remark 1. The $k = 12$ case is due to Ono [13, 14]. In this case, $a_{1,12}(m) = \tau(m)$, and the theorem provides a criterion for verifying Lehmer's Conjecture. The result for weights $k \neq 12$ follows from work of the first author [5].

Therefore, our main result extends the results of Theorem 1.1 to other levels $N \in S$.

Theorem 1.2. *Assume the notation above.*

(1) *Let $N = 2$, and suppose that m is a positive integer. Then $a_{2,8}(m) = 0$ if and only if $A_{2,8,G,m}(x) = 0$, where (G, x) is given by the following table:*

G	x
$E_6(z)$	$-2^6,\ 2^9$
$E_6(2z)$	$2^3,\ -2^6$
$E_{2,6,\infty}(z)$	-2^6
$E_4(z)E^*_{2,2}(z)$	$-2^6,\ -2^8$
$E_4(2z)E^*_{2,2}(z)$	$2^4,\ -2^6$

(2) *Let $N = 3$, and suppose that m is a positive integer. Then $a_{3,6}(m) = 0$ if and only if $A_{3,6,G,m}(x) = 0$, where (G, x) is given by the following table:*

G	x
$E_4(z)$	$-3^3,\ -3^5$
$E_4(3z)$	$-3,\ -3^3$
$E_{3,4,\infty}(z)$	-3^3

Remark 2. Analogous results hold for $(N, k) \in \{(2, 10), (3, 8)\}$ using the identities $f_{2,10}(z) = f_{2,8}(z) \cdot E^*_{2,2}(z)$ and$f_{3,8}(z) = f_{3,6}(z) \cdot E^*_{2,2}(z)$; see Section 2.

Remark 3. Let $\omega := \frac{-1+\sqrt{-3}}{2}$. The arguments x in the theorem are precisely the rational integer values of $\phi_N(\tau)$ at elliptic points of $\Gamma_0(N)$ or points $\tau \in \{i, \omega, i/N, \omega/N\}$. Equivalently, the x values arise as the values of $\phi_N(\tau)$ at zeros of $G(z)$. In particular, the valence formula shows that $G_2(z) \in M_6(\Gamma_0(2))$ must have a zero at $\frac{1+i}{2}$, a representative of the unique elliptic point for $\Gamma_0(2)$, and that $G_3(z) \in M_4(\Gamma_0(2))$ must have a zero at $\frac{2+\omega}{3}$, a representative of the unique elliptic point for $\Gamma_0(3)$. Observing that $\phi_2\left(\frac{1+i}{2}\right) = -2^6$ and that $\phi_3\left(\frac{2+\omega}{3}\right) = -3^3$, it follows that part (1) of the theorem holds for all pairs $(G_2(z), -2^6)$, while part (2) of the theorem holds for all pairs $(G_3(z), -3^3)$.

We note that there may be points τ of this type for which $\phi_N(\tau)$ is not a rational integer. For example, when $N = 3$, one can show (see Section 2) that while $\phi_3\left(\frac{2+\omega}{3}\right)$, $\phi_3(\omega)$, and $\phi_3\left(\frac{\omega}{2}\right)$ are rational integers, we have

$$\phi_3(i) = -9 + 6\sqrt{3} = 3(-3 + 2\sqrt{3}), \quad \phi_3\left(\frac{i}{3}\right) = 243 - 162\sqrt{3} = 3^4(3 - 2\sqrt{3}). \tag{1.16}$$

Moreover, for $N \in \{5, 7, 13\}$, none of the values of $\phi_N(\tau)$ at such points are rational integers. Nevertheless, one could derive further results of the type in Theorem 1.2 by studying $A_{G,N,k,m}(x)$ at algebraic integral arguments x which arise as values of $\phi_N(\tau)$ at quadratic irrationalities in $\mathbb{H}$. For simplicity and aesthetics, we do not pursue further investigation of examples of this type.

Here we list examples of the polynomials $A_{N,k,G,m}(x)$ above. For $(N,k)=(2,8)$ and $m=3$ we have

$$\begin{aligned}
A_{2,8,E_6(z),3}(x) &= x^3-432x^2-39924x-\frac{9010432}{17}\,,\\
A_{2,8,E_6(2z),3}(x) &= x^3+72x^2-396x-\frac{133984}{17}\,,\\
A_{2,8,E_{2,6,\infty}(z),3}(x) &= x^3+80x^2+1036x+\frac{6912}{17}\,,\\
A_{2,8,E_4(z)E^*_{2,2}(z),3}(x) &= x^3+336x^2+21516x+\frac{4515584}{17}\,,\\
A_{2,8,E_4(2z)E^*_{2,2}(z),3}(x) &= x^3+96x^2+2316x+\frac{288704}{17}\,.
\end{aligned}$$

Similarly, for $(N,k)=(3,6)$ and $m=2$ we have

$$\begin{aligned}
A_{3,6,E_4(z),2}(x) &= x^2+264x+\frac{65223}{13}\,,\\
A_{3,6,E_4(3z),2}(x) &= x^2+24x-\frac{297}{13}\,,\\
A_{3,6,E_{3,4,\infty}(z),2}(x) &= x^2+21x-\frac{1116}{13}\,.
\end{aligned}$$

Note that in the examples above, all coefficients are integers except for the constant terms. If $G(z)$ has integer coefficients, it follows from (1.15) that all coefficients of $A_{N,k,G,m}(x)$ must be integral with the possible exception of the constant term. The integrality of the constant term depends on $\beta_{N,k}(m)$, the mth coefficient of $E_{N,k,\infty}(z)$. Thus, a necessary condition for $a_{N,k}(m)$ to vanish is that $\beta_{N,k}(m)$ be an integer. Consequently, if p is a prime for which $a_{2,8}(p)=0$, then (1.5) implies that $p\equiv -1 \pmod{17}$. Similarly, if p is a prime for which $a_{3,6}(p)=0$, then we must have $p\equiv -1 \pmod{13}$. These observations are consistent with those following from the congruences (1.8).

The outline of the paper is as follows. In Section 2, we prove Theorem 1.2. We do this directly using facts about modular forms. In Section 3, we describe an alternate approach to the proof using harmonic weak Maass forms.

2. Proof of Theorem 1.2

Let $\tau\in\mathbb{H}$. In [4], Asai, Kaneko, and Ninomiya proved that

$$\sum_{n=0}^{\infty} J_{1,n}(\tau)q^n = -\frac{\theta(j(z)-j(\tau))}{j(z)-j(\tau)} = \frac{E_{14}(z)}{\Delta(z)}\cdot\frac{1}{j(z)-j(\tau)}. \tag{2.1}$$

This formula is equivalent to the denominator formula for the Monster Lie Algebra and has connections to other "Moonshine" phenomena; for details, see [8]. Since $j(i)=1728$ and $j(\omega)=0$, equation (2.1) has the following specializations:

$$H_{1,1728}(z)=\sum_{n=0}^{\infty}p_{1,n}(1728)q^n=\sum_{n=0}^{\infty}J_{1,n}(i)q^n=\frac{E_6(z)}{E_4(z)}, \tag{2.2}$$

$$H_{1,0}(z)=\sum_{n=0}^{\infty}p_{1,n}(0)q^n=\sum_{n=0}^{\infty}J_{1,n}(\omega)q^n=\frac{E_8(z)}{E_6(z)}. \tag{2.3}$$

Now, let $N \neq 1$ in S. The generating function for $J_{N,n}(\tau)$ has a formula similar to (2.1), which we now describe. With $\phi_N(z)$ as in (1.9), Ahlgren [3] proved the formula

$$\sum_{n=1}^{\infty} j_{N,n}(\tau)q^n = \frac{\theta(\phi_N(z)-\phi_N(\tau))}{\phi_N(z)-\phi_N(\tau)} - E^*_{N,2}(z) = E^*_{N,2}(z)\left(\frac{\phi_N(z)}{\phi_N(z)-\phi_N(\tau)} - 1\right). \tag{2.4}$$

Let $c_{N,n}$ be as in (1.10). An application of the Residue Theorem to $(j_{N,n}(z) - c_{N,n}) \cdot E_{N,2,\infty}(z) \in M_2^!(\Gamma_0(N))$ shows that

$$c_{N,n} = \frac{24}{N-1} \cdot \sigma_{\chi_N^{\mathrm{triv}},1}(n).$$

From (1.3), (1.11), and (2.4), it follows that

$$\sum_{n=0}^{\infty} J_{N,n}(\tau)q^n = \frac{\theta(\phi_N(z)-\phi_N(\tau))}{\phi_N(z)-\phi_N(\tau)} = \frac{E^*_{N,2}(z)\phi_N(z)}{\phi_N(z)-\phi_N(\tau)}. \tag{2.5}$$

In this setting, we seek analogues of (2.2) and (2.3). When $N = 2$, we consider

$$j(z) = \frac{(\phi_2(z)+256)^3}{\phi_2(z)^2}, \tag{2.6}$$

$$j(2z) = \frac{(\phi_2(z)+16)^3}{\phi_2(z)} \tag{2.7}$$

for this purpose. For the remaining values of $N \neq 1$ in S, we record analogous expressions

$$j(z) = \frac{\psi_{N,1}(\phi_N(z))}{\phi_N(z)^N}, \qquad j(Nz) = \frac{\psi_{N,2}(\phi_N(z))}{\phi_N(z)} \tag{2.8}$$

in the following table:

N	$\psi_{N,1}(x)$	$\psi_{N,2}(x)$
3	$(x+243)^3(x+27)$	$(x+3)^3(x+27)$
5	$(x^2+250x+3125)^3$	$(x^2+10x+5)^3$
7	$(x^2+13x+49)(x^2+245x+2401)^3$	$(x^2+13x+49)(x^2+5x+1)^3$
13	$(x^2+5x+13)$ $\times(x^4+247x^3+3380x^2+1183x+28561)^3$	$(x^2+5x+13)$ $\times(x^4+7x^3+20x^2+19x+1)^3$

Returning to $N = 2$, we compute the value of ϕ_2 at elliptic points of $\Gamma_0(2)$ and at points $\tau \in \{i, \omega, i/2, \omega/2\}$. We observe that $\Gamma_0(2)$ has a single elliptic point of order 2, represented by $(i+1)/2$, and that $j\left(\frac{i+1}{2}\right) = j(i) = 1728$. From (2.6), we then conclude that $x = \phi_2\left(\frac{i+1}{2}\right)$ and $x = \phi_2(i)$ are roots of the polynomial

$$(x+256)^3 - 1728x^2 = (x+64)(x-512)^2 = 0.$$

Using the q-expansion for $\phi_2(z)$, we find that

$$\phi_2\left(\frac{1+i}{2}\right) = -64, \qquad \phi_2(i) = 512. \tag{2.9}$$

Similarly, if we let $z = i/2$ in (2.7), we deduce that

$$\phi_2\left(\frac{i}{2}\right) = 8.$$

Alternatively, one may compute the last value from the identity

$$\phi_N(-1/z) = \frac{N^{12/(N-1)}}{\phi_N(z/N)},$$

which follows from the transformation formula for $\eta(z)$ and the fact that $j(z) = j(-1/z)$. We also substitute $j(\omega) = 0$ in (2.6) and (2.7) to obtain

$$\phi_2(\omega) = -256, \qquad \phi_2\left(\frac{\omega}{2}\right) = -16.$$

For each of the preceding special values, one may use standard techniques to express $\phi_2(z) - \phi_2(\tau)$ as a ratio of Eisenstein series. For example, we have

$$\phi_2(z) + 64 = \phi_2(z) - \phi_2\left(\frac{1+i}{2}\right) = \frac{E_{2,2}^*(z)^2}{E_{2,4,0}(z)}.$$

In view of such identities, we conclude the following analogues of (2.2) and (2.3) for $N = 2$.

Lemma 2.1. *Assume the notation in* (1.1), (1.3), (1.4), (1.11), (1.12), *and* (1.13). *Then the following identities hold:*

$$H_{2,8}(z) = \sum_{n=0}^{\infty} p_{2,n}(8)q^n = \sum_{n=0}^{\infty} J_{2,n}\left(\frac{i}{2}\right)q^n = \frac{E_{2,6,\infty}(z)E_{2,2}^*(z)}{E_6(2z)},$$

$$H_{2,-16}(z) = \sum_{n=0}^{\infty} p_{2,n}(-16)q^n = \sum_{n=0}^{\infty} J_{2,n}\left(\frac{\omega}{2}\right)q^n = \frac{E_{2,6,\infty}(z)}{E_4(2z)},$$

$$H_{2,-64}(z) = \sum_{n=0}^{\infty} p_{2,n}(-64)q^n = \sum_{n=0}^{\infty} J_{2,n}\left(\frac{1+i}{2}\right)q^n = \frac{E_{2,6,\infty}(z)}{E_{2,2}^*(z)^2},$$

$$H_{2,-256}(z) = \sum_{n=0}^{\infty} p_{2,n}(-256)q^n = \sum_{n=0}^{\infty} J_{2,n}(\omega)q^n = \frac{E_{2,6,\infty}(z)}{E_4(z)},$$

$$H_{2,512}(z) = \sum_{n=0}^{\infty} p_{2,n}(512)q^n = \sum_{n=0}^{\infty} J_{2,n}(i)q^n = \frac{E_{2,6,\infty}(z)E_{2,2}^*(z)}{E_6(z)}.$$

We next consider $N = 3$ and similarly calculate $\phi_3(z)$ at elliptic points of $\Gamma_0(3)$ and points $\tau \in \{i, \omega, i/3, \omega/3\}$. The congruence subgroup $\Gamma_0(3)$ has a unique elliptic point, represented by $(2+\omega)/3$. We find that $j\left(\frac{2+\omega}{3}\right) = j(\omega) = 0$. Substituting this information in the expressions for $j(z)$ and $j(3z)$ in terms of $\phi_3(z)$, we obtain

$$\phi_3(\omega) = -243, \quad \phi_3\left(\frac{2+\omega}{3}\right) = -27, \quad \phi_3\left(\frac{\omega}{3}\right) = -3. \tag{2.10}$$

On the other hand, (1.16) shows that $\phi_3(i)$ and $\phi_3(i/3)$ are algebraic integers, but not rational integers.

We further require $g_3(z)$, a modular form which transforms with respect to the character $\chi_{-3}(\cdot) := \left(\frac{-3}{\cdot}\right)$:

$$g_3(z) := \frac{\eta(z)^9}{\eta(3z)^3} \in M_3(\Gamma_0(3), \chi_{-3}). \tag{2.11}$$

For $\tau \in \left\{\frac{2+\omega}{3}, \omega, \frac{\omega}{3}\right\}$, we may express $\phi_3(z) - \phi_3(\tau)$ as a rational function of Eisenstein series with rational coefficients, from which we conclude the following lemma.

Lemma 2.2. *Assume the notation in* (1.1), (1.3), (1.4), (1.11), (1.12), (1.13), *and* (2.11). *We have the following relations:*

$$\begin{aligned}
H_{3,-3}(z) &= \sum_{n=0}^{\infty} p_{3,n}(-3)q^n = \frac{E^*_{3,2}(z)E_{3,4,\infty}(z)}{E_4(3z)},\\
H_{3,-27}(z) &= \sum_{n=0}^{\infty} p_{3,n}(-27)q^n = \frac{g_3(z)^2}{E_{3,4,\infty}(z)},\\
H_{3,-243}(z) &= \sum_{n=0}^{\infty} p_{3,n}(-243)q^n = \frac{E^*_{3,2}(z)E_{3,4,\infty}(z)}{E_4(z)}.
\end{aligned}$$

We now use Lemmas 2.1 and 2.2 to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $(N, k) = (2, 8)$. For the pairs $(G(z), x)$ in part (1) of the theorem, (1.14) together with Lemma 2.1 imply that

$$\sum_{m=1}^{\infty} A_{2,8,G(z),m}(x)q^m = H_{2,x}(z) \cdot G(z) - E_{2,8,\infty}(z) \in M_8(\Gamma_0(2)).$$

Since $M_8(\Gamma_0(2))$ has dimension 3, one may now readily deduce for each such pair $(G(z), x)$ that there is a constant $c_{G,x}$, as in the following table, for which

$$\sum_{m=1}^{\infty} A_{2,8,G(z),m}(x)q^m = c_{G,x} \cdot f_{2,8}(z).$$

Note that this constant must be $c_{G,x} = A_{2,8,G(z),1}(x)$. Hence, we see for all $m \geq 1$ that $a_{2,8}(m)$ is a non-zero multiple of $A_{2,8,G(z),m}(x)$.

$G(z)$	x	$c_{G,x}$
$E_6(z)$	512	$\frac{576}{17}$
$E_6(2z)$	8	$\frac{576}{17}$
$E_4(z)E^*_{2,2}(z)$	-256	$\frac{576}{17}$
$E_4(2z)E^*_{2,2}(z)$	-16	$\frac{576}{17}$
$E_6(z)$	-64	$-\frac{9216}{17}$
$E_6(2z)$	-64	$-\frac{648}{17}$
$E_4(z)E^*_{2,2}(z)$	-64	$\frac{3840}{17}$
$E_4(2z)E^*_{2,2}(z)$	-64	$-\frac{240}{17}$
$E_{2,6,\infty}(z)$	-64	$-\frac{512}{17}$

Part (2) of Theorem 1.2 follows similarly. Let $(N,k)=(3,6)$. For each pair $(G(z),x)$ in part (2) of the theorem, (1.14) and Lemma 2.2 imply that there is a constant $c_{G,x}=A_{3,6,G(z),1}(x)$ for which

$$\sum_{m=1}^{\infty} A_{3,6,G(z),m}(x)q^m = c_{G,x}\cdot f_{3,6}(z).$$

We give the $c_{G,x}$ below. The theorem follows.

$G(z)$	x	$c_{G,x}$
$E_4(z)$	-243	$\frac{108}{13}$
$E_4(3z)$	-3	$\frac{108}{13}$
$E_4(z)$	-27	$\frac{2916}{13}$
$E_4(3z)$	-27	$\frac{-204}{13}$
$E_{3,4.\infty}(z)$	-27	$-\frac{243}{13}$

□

Remark 4. One may combine the objects we consider in various ways to obtain further identities. For example, we have:

(1) $f_{2,8}(z)=H_{2,-64}(z)\cdot E_{2,8,0}(z)$.
(2) $f_{3,6}(z)=H_{3,-27}(z)\cdot E_{3,6,0}(z)$.
(3) $H_{2,512}(z)\cdot E_6(z)=H_{2,8}(z)\cdot E_6(2z)$.
(4) $H_{2,-256}(z)\cdot E_4(z)=H_{2,-16}(z)\cdot E_4(2z)$.
(5) $H_{3,-243}(z)\cdot E_4(z)=H_{3,-3}(z)\cdot E_4(3z)$.

Observe that identities (3) and (4) follow from the tables in the proof of Theorem 1.2.

Remark 2 following Theorem 1.2 states that analogous results hold for $(N,k)\in\{(2,10),(3,8)\}$. For $(N,k)=(2,10)$, these results follow from the proof of Theorem 1.2 and the identities

$$E^*_{2,2}(z)f_{2,8}(z)=f_{2,10}(z),\quad E^*_{2,2}(z)E_{2,8,\infty}(z)=E_{2,10,\infty}(z)+\frac{11520}{527}\cdot f_{2,10}(z). \tag{2.12}$$

For example, the proof of Theorem 1.2 part (1) for $(G(z), x) = (E_6(2z), 8)$ gives

$$H_{2,8}(z) \cdot E_6(2z) - E_{2,8,\infty}(z) = \frac{576}{17} \cdot f_{2,8}(z).$$

Multiplying both sides by $E^*_{2,2}(z)$ and using (2.12), we conclude that

$$\sum_{m=1}^{\infty} A_{2,10,E^*_{2,2}(z)E_6(2z),m}(8)q^m = H_{2,8}(z) \cdot E^*_{2,2}(z)E_6(2z) - E_{2,10,\infty}(z) = \frac{1728}{31} \cdot f_{10,2}(z).$$

It follows that $a_{2,10}(m) = 0$ if and only if $A_{2,10,E^*_{2,2}(z)E_6(2z),m}(8) = 0$. Similar results hold for $(N, k) = (3, 8)$ using the proof of Theorem 1.2 and the identities

$$E^*_{3,2}(z)f_{3,6}(z) = f_{3,8}(z), \quad E^*_{3,2}(z)E_{3,6,\infty}(z) = E_{3,8,\infty}(z) + \frac{6804}{533} \cdot f_{3,8}(z).$$

Furthermore, one can derive variants of the results of Theorem 1.2 in certain cases. For example, when $N = 5$, we observe that $\Gamma_0(5)$ has elliptic points of order 2 represented by $\tau \in \{\frac{2+i}{5}, \frac{3+i}{10}\}$. For such τ, we note that $j(\tau) = j(5\tau) = 1728$. With $\psi_{5,1}(x)$ and $\psi_{5,2}(x)$ as in (2.8), we determine that $1728x^5 = \psi_{5,1}(x)$ and that $1728x = \psi_{5,2}(x)$, and we conclude using Fourier expansions that

$$\phi_5\left(\frac{2+i}{5}\right) = -11 - 2i, \qquad \phi_5\left(\frac{3+i}{10}\right) = -11 + 2i.$$

To pursue this example, we define the auxiliary form

$$g_5(z) := \frac{\eta^{10}(z)}{\eta^2(5z)} = 1 + O(q) \in M_4(\Gamma_0(5)).$$

Using the identities

$$E^*_{5,2}(z)\phi_5(z) = \frac{E_{2,5,\infty}(z)g_5(z)}{f_{5,4}(z)}, \quad \phi_5(z)^2 + 22\phi_5(z) + 125 = \frac{E^*_{5,2}(z)^2 g_5(z)}{f^2_{5,4}(z)},$$

we find that

$$H_{5,-11+2i}(z)H_{5,-11-2i}(z) = g_5(z)$$

and that

$$H_{5,-11+2i}(z)H_{5,-11-2i}(z) - E_{5,4,\infty}(z) = -\frac{125}{13} \cdot f_{5,4}(z),$$
$$H_{5,-11+2i}(z)H_{5,-11-2i}(z) \cdot E^*_{5,2}(z) - E_{5,6,\infty}(z) = -\frac{125}{31} \cdot f_{5,6}(z).$$

3. An alternate viewpoint: Harmonic Weak Maass Forms

In [13], Ono gives two proofs of Theorem 1.1 for weight $k = 12$. The first follows from manipulation of (2.2) and (2.3) via facts on classical modular forms. The second follows from interplay between modular forms and harmonic weak Maass forms. Using facts in [5], both methods extend to prove the remaining cases of Theorem 1.1. In this section we sketch the main ideas in the proof of Theorem 1.2 from the viewpoint of harmonic weak Maass forms. We begin with some definitions; throughout we assume the notation from Section 1.

We recall the definition of a harmonic weak Maass form. Let $z = x + iy \in \mathbb{H}$. We require the weight k hyperbolic Laplacian, denoted by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

A harmonic weak Maass form of weight k on $\Gamma_0(N)$ is a smooth function f on $\mathbb{H}$ satisfying:

(i) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have $(f|_k M)(z) = f(z)$.

(ii) $\Delta_k f = 0$.

(iii) There is a polynomial $P_f = \sum_{n\le 0} c_f^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. One requires analogous conditions at all cusps.

For a fixed cusp, one calls the corresponding polynomial P_f in (iii) the principal part of the expansion at the cusp. We denote the vector space of forms satisfying (i)-(iii) by $H_k(\Gamma_0(N))$. For details on harmonic weak Maass forms, see [16] for example.

Let $k \geq 2$ and $N \geq 1$ be integers. Important examples of forms in $H_{2-k}(\Gamma_0(N))$ include non-holomorphic Poincaré series, as we now describe. For details on series of this type, see [6] or [7]; for details on the special functions required to define these series, see [1] or [2]. Let $s \in \mathbb{C}$, let $y \in \mathbb{R} - \{0\}$, let $e(x) := e^{2\pi ix}$, and let $M_{\nu,\mu}$ be the usual M-Whittaker function with parameters μ, ν. We define

$$\mathcal{M}_s(y) := |y|^{\frac{k}{2}-1}M_{\operatorname{sgn}(y)\left(1-\frac{k}{2}\right),s-\frac{1}{2}}(|y|),$$

and we define

$$\phi_s(z) := \mathcal{M}_s(4\pi y)e(x).$$

We let $\Gamma_0(N)_\infty$ denote the stabilizer of ∞ in $\Gamma_0(N)$. For matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we define the slash operator $|_k M$ (see [11] for example) on functions $f(z)$ in the usual way, and we define $\chi_N^{\mathrm{triv}}(M) := \chi_N^{\mathrm{triv}}(d)$. We may now define the non-holomorphic Poincaré series

$$F_{N,2-k}(z) := \frac{1}{(k-1)!}\sum_{M\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)}\left(\chi_N^{\mathrm{triv}}(M)^{-1}\phi_{k/2}\left(-z\right)|_{2-k}M\right) \in H_{2-k}(\Gamma_0(N)). \tag{3.1}$$

To describe q-expansions for $F_{N,2-k}(z)$, we require $\Gamma(a,x)$, the incomplete Gamma-function. One may canonically decompose $F_{N,2-k}(z)$ as a sum of holomorphic and non-holomorphic parts, writing

$$F_{N,2-k}(z) = F^+_{N,2-k}(z) + F^-_{N,2-k}(z),$$

with

$$F^+_{N,2-k}(z) := q^{-1} + \sum_{n=0}^{\infty} a^+(n)q^n,$$

$$F^-_{N,2-k}(z) := -\frac{1}{(k-2)!}\Gamma(k-1,4\pi y)q^{-1} + \sum_{n=1}^{\infty} a^-(n)\Gamma(k-1,4\pi ny)q^{-n};$$

exact formulae for the $a^+(n)$ and $a^-(n)$ are given in [6]. We note that for $N \in S$, we have

$$a^+(0) = \begin{cases} -\frac{2k}{B_k(N^k-1)} & N = 2,3,5,7,13, \\ \frac{2k}{B_k} & N = 1. \end{cases}$$

We now describe the interplay between newforms $f_{N,k}(z)$ as in (1.7) and harmonic weak Maass forms $F_{N,2-k}(z)$ as in (3.1). We require the differential operator

$$\xi_{2-k} := 2iy^{2-k} \cdot \overline{\frac{\partial}{\partial \overline{z}}}.$$

This operator satisfies $\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to S_k(\Gamma_0(N))$. If $||\cdot||$ denotes the Petersson product on $S_k(\Gamma_0(N))$, then one can show that $\xi_{2-k}(F_{N,2-k}(z)) = ||f_{N,k}(z)||^{-2} \cdot f_{N,k}(z)$. As such, we say that $F_{N,k}(z)$ is *good* for $f_{N,k}(z)$. For the complete definition of what it means for a form in $H_{2-k}(\Gamma_0(N))$ to be good for a normalized newform in $S_k(\Gamma_0(N))$, see [10].

Remark 5. More generally, suppose that $F(z) = F^+(z) + F^-(z) \in H_{2-k}(\Gamma_0(N))$ is good for a newform $f(z) \in S_k(\Gamma_0(N))$ and that $p \nmid N$ is prime. Theorem 1.4 of [10] gives the following necessary condition for the vanishing of coefficients $a_f(p)$ of $f(z)$ in terms of algebraicity of coefficients $a^+(n)$ of $F^+(z)$. Let $\mathrm{ord}_p(n)$ denote the highest power of p dividing the integer n. If $a_f(p) = 0$, then for all n with $\mathrm{ord}_p(n)$ odd, $a^+(n)$ is algebraic. Theorem 1.3 of [10] asserts that if $f(z)$ has complex multiplication, then for all n, $a^+(n)$ is algebraic. A question of Ono [16] asks whether the converse is true.

We now describe how the polynomials $A_{N,k,G,\ell}(z)$ in (1.14) arise from harmonic weak Maass forms. Recall that the forms $f_{N,k}(z)$ have Fourier coefficients $a_{N,k}(n)$. Let $T_{N,k}(m)$ denote the usual Hecke operator of weight k, level N, and index m. Since the $f_{N,k}(z)$ are newforms, they satisfy, for all primes $\ell \neq N$,

$$f_{N,k}(z) \mid T_{N,k}(\ell) = a_{N,k}(\ell) f_{N,k}(z).$$

Therefore, for a fixed prime $\ell \neq N$, we define

$$L_{N,k,\ell}(z) := \ell^{k-1}(F_{N,2-k}(z) \mid T_{N,2-k}(\ell) - \ell^{1-k} a_{N,k}(\ell) F_{N,2-k}(z)).$$

Arguing as in [5, Proposition 3.1] or [9, Section 7], one can show, since $F_{N,k}(z)$ is good for $f_{N,k}(z)$, that $F^-_{N,2-k}(z)$ is an eigenform for $T_{N,2-k}(\ell)$ with eigenvalue $\ell^{1-k} a_{N,k}(\ell)$. It follows that $L_{N,k,\ell}(z) \in M^!_{2-k}(\Gamma_0(N))$ and that

$$\begin{aligned} L_{N,k,\ell}(z) &= \ell^{k-1}(F^+_{N,2-k}(z) \mid T_{N,2-k}(\ell) - \ell^{1-k} a_{N,k}(\ell) F^+_{N,2-k}(z)) \\ &= q^{-\ell} - a_{N,k}(\ell) q^{-1} + a^+(0)(\sigma_{k-1}(\ell) - a_{N,k}(\ell))) \\ &\quad + \sum_{n=1}^{\infty} \left(\ell^{k-1} a^+(\ell n) - a_{N,k}(\ell) a^+(n) + a^+\left(\frac{n}{\ell}\right)\right) q^n. \end{aligned}$$

In order to obtain a convenient closed-form expression for $L_{N,k,\ell}(z)$, we require an auxiliary modular form $G(z)$ as in Section 1:

$$G(z) = \sum_{n=0}^{\infty} b_G(n) q^n = 1 + \cdots \in M_{k-2}(\Gamma_0(N)).$$

By construction, the forms $L_{N,k,\ell}(z)$ and $L_{N,k,\ell}(z) \cdot G(z) \in M^!_0(\Gamma_0(N))$ have non-vanishing principal parts at infinity, but vanish at the cusp zero. We compute

$$L_{N,k,\ell}(z) \cdot G(z) = \sum_{n=0}^{\ell} b_G(n) q^{n-\ell} - a_{N,k}(\ell) \sum_{n=0}^{1} b_G(n) q^{n-1} + a^+(0)(\sigma_{k-1}(\ell) - a_{k,N}(\ell)) + O(q).$$

With $\phi_N(z)$ as in (1.9) and $A_{N,k,G,m}(x)$ as in (1.14), we observe that

$$A_{N,k,G,\ell}(\phi_N(z)) - a_{N,k}(\ell)A_{G,N,k,1}(\phi_N(z))$$

and $L_{N,k,\ell}(z)\cdot G(z)$ have the same principal part at infinity. Since both forms are holomorphic at zero, their difference is

$$L_{N,k,\ell}(z)\cdot G(z) - (A_{N,k,G,\ell}(\phi_N(z)) - a_{N,k}(\ell)A_{N,k,G,1}(\phi_N(z))) \in M_0(\Gamma_0(N)).$$

Hence, it is constant. In particular, the difference vanishes at infinity and is therefore equal to zero, from which we conclude that

$$L_{N,k,\ell}(z) = \frac{A_{N,k,G,\ell}(\phi_N(z)) - a_{N,k}(\ell)A_{N,k,G,1}(\phi_N(z))}{G(z)}. \tag{3.2}$$

Now, recall that $L_{N,km,\ell}(z) \in M^!_{2-k}(\Gamma_0(N))$. Hence, if $G(z) \in M_{k-2}(\Gamma_0(N))$ vanishes at $\tau \in \mathbb{H}$, then $A_{N,k,G,\ell}(\phi_N(z)) - a_{N,k}(\ell)A_{N,k,G,1}(\phi_N(z))$ must also vanish. Using the valence formula, we recover the results in Theorem 1.2; see, for example, Remark 3.

For $(N,k) \in \{(2,10),(3,8),(5,4),(5,6)\}$, the valence formula reveals no information about vanishing of $G(z) \in M_{k-2}(\Gamma_0(N))$ at elliptic points. It does, however, yield information when $(N,k) = (7,4)$. In this case, $M_2(\Gamma_0(7)) = \mathbb{C}E^*_{7,2}(z)$, so it suffices to consider $G(z) = E^*_{7,2}(z)$, a form with integer coefficients. The valence formula reveals that $E^*_{7,2}(z)$ vanishes at one or both representatives $\tau \in \{\frac{-2+\omega}{7}, \frac{3+\omega}{7}\}$ of the elliptic points for $\Gamma_0(7)$. By (2.8), the corresponding values $\phi_7(\tau)$ must be common roots of the polynomials $\psi_{7,1}(x)$ and $\psi_{7,2}(x)$. We find these common roots to be

$$x_\pm = \frac{-13 \pm 3i\sqrt{3}}{2}.$$

Note that they are not rational integers. As they are complex conjugates, it follows from (1.13) that the Fourier coefficients of $H_{7,x_\pm}(z)$ are complex conjugates. Therefore, the numerator in (3.2) vanishes at both elliptic points. We conclude that

$$A_{7,4,G,\ell}\left(\frac{-13 \pm 3i\sqrt{3}}{2}\right) = a_{7,4}(\ell)A_{7,4,G,1}\left(\frac{-13 \pm 3i\sqrt{3}}{2}\right) = \left(\frac{8}{5} \pm \frac{3\sqrt{3}}{2}i\right)a_{7,4}(\ell).$$

Hence, for $(N,k) = (7,4)$, we obtain a vanishing result similar to Theorem 1.2.

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Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: `boylan@math.sc.edu`

Department of Mathematics, Bucknell University, Lewisburg, PA
E-mail address: `Sharon.Garthwaite@bucknell.edu`

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: `webbjj3@mailbox.sc.edu`