CONGRUENCES FOR 2^t-CORE PARTITION FUNCTIONS

Matthew Boylan

July 23, 2000.

1 Introduction and Statement of Results

Here, we find for any integer $t \geq 1$, infinitely many congruences for the 2^t-core partition functions $\pmod 2$. One can use Ferrers-Young diagrams to investigate $p(n)$, the partition function. If $\Lambda = \lambda_1 \geq \cdots \geq \lambda_k$ is a partition of n, then the Ferrers-Young diagram of Λ is the following staircase arrangement of nodes in k rows:

$$\begin{array}{llllll} \bullet & \bullet & \ldots & \bullet & \bullet & \quad \lambda_1 \text{ nodes} \\ \bullet & \bullet & \ldots & \bullet & & \quad \lambda_2 \text{ nodes} \\ \cdot & & & & & \\ \cdot & & & & & \\ \bullet & \ldots & \bullet & & & \quad \lambda_k \text{ nodes} \end{array}$$

Label the nodes (i, j), as if they were matrix entries. We associate a geometric object called a hook to each node. The hook associated to the (i, j) node consists of the nodes directly to the right of the specified node, the nodes directly below it, and the node itself. If λ_j' denotes the number of nodes in column j, the hook number $H(i, j)$ of the (i, j) node is defined by:

$$H(i, j) := \lambda_i + \lambda_j' - i - j + 1$$

We now define a t-core partition of a positive integer n.

Definition. *If t is a positive integer, then Λ is a t-core partition if none of the hook numbers associated to the Ferrers-Young diagram of Λ is a multiple of t. We denote the number of t-core partitions of n by $a_t(n)$.*

Example. Let Λ be the partition of 8 defined by $\Lambda = 4, 3, 1$. Then the Ferrers-Young diagram of Λ is:

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & \bullet & \\ 3 & \bullet & & & \end{array}$$

The hook numbers are: $H(1,1)=6$, $H(1,2)=4$, $H(1,3)=3$, $H(1,4)=1$, $H(2,1)=4$, $H(2,2)=2$, $H(2,3)=1$, and $H(3,1)=1$. Hence, it $t\notin\{1,2,3,4,6\}$, then Λ is a t-core partition of 8.

We are interested in 2^t-core partition functions in light of some recent conjectures and theorems of Hirschhorn, Kolitsch, and Sellers concerning the parity of these functions in certain arithmetic progressions. One such conjecture is the following:

Conjecture. *(Hirschhorn, Sellers [2]) If t and n are positive integers with $t\geq 2$, then for $k=0,2$,*

$$a_{2^t}\left(\frac{3^{2^{t-1}-1}(24n+8k+7)-\frac{4^t-1}{3}}{8}\right)\equiv 0 \pmod 2.$$

This conjecture has been proved by Hirschhorn, Kolitsch, and Sellers for $t=2,3,4$. In our investigation, we observe that such congruences for 2^t-core partition functions $\pmod 2$ can be understood by analyzing the action of the Hecke operators T_p, where p is an odd prime, on $S_{4^{t+1}-4} \pmod 2$, the space of cusp forms of weight $4^{t+1}-4$ with integer coefficients on $SL_2(\mathbb{Z}) \pmod 2$. Using this observation, we prove the following two general theorems which hold for all t. We then calculate some examples.

Theorem 1.1. *For any positive integer $t\geq 1$, if $p_1,\ldots,p_{\frac{4^t-1}{3}}$ are distinct odd primes, then*

$$a_{2^t}\left(\frac{p_1\cdots p_{\frac{4^t-1}{3}}N-\frac{4^t-1}{3}}{8}\right)\equiv 0 \pmod 2$$

for every N with $\gcd(N,\prod p_i)=1$.

Theorem 1.2. *If p is an odd prime, then*

$$a_{2^t}\left(\frac{p^{2^{2t-j}-1}N-\frac{4^t-1}{3}}{8}\right)\equiv 0 \pmod 2,$$

whenever $\gcd(N,p)=1$, where

$$j=\begin{cases} 4 & \text{if } p\equiv 1 \pmod 8,\ t\geq 3\\ 3 & \text{if } p\equiv 1 \pmod 8,\ t=2, \text{ or } p\equiv 3 \pmod 8,\ t\geq 2, \text{ or } p\equiv 5,7 \pmod 8,\ t\geq 3\\ 2 & \text{if } p\equiv 5,7 \pmod 8,\ t=2\\ 1 & \text{for any odd prime } p, \text{ any positive integer } t.\end{cases}$$

Remark. When $p = 3$, $t = 6$, and $k = 0$ or 2, Theorem 1.2 yields

$$a_{64}\left(\frac{3^{511}(24n+8k+7)-1365}{8}\right) \equiv 0 \pmod{2},$$

whereas the Hirschhorn-Sellers Conjecture yields

$$a_{64}\left(3^{31}(24n+8k+7)-1365\right) \equiv 0 \pmod{2}.$$

Hence, while Theorem 1.2 does not give the full Hirschhorn-Sellers Conjecture, it gives congruences very similar to those appearing in the Conjecture for every positive integer t and every odd prime p, not only $p = 3$.

2. The proof of Theorems 1.1 and 1.2.

We first establish the relationship between 2^t-core partition functions and modular forms. The two modular forms of interest to us are Dedekind's eta function $\eta(z)$, defined by the infinite product

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$, and Ramanujan's delta function, $\Delta(z)$, the unique normalized weight 12 cusp form on $SL_2(\mathbb{Z})$, given by

$$\Delta(z) = \eta^{24}(z).$$

In what follows, we denote by S_k the space of cusp forms of weight k with integer coefficients on $SL_2(\mathbb{Z})$. Moreover, let $S_k \pmod{2}$ denote the space of modular forms in S_k with integer coefficients, reduced modulo 2. From [1], the generating function for $a_t(n)$ is given by the following infinite product:

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{nt})^t}{(1-q^n)}. \tag{1}$$

From the definitions of $\eta(z)$ and $\Delta(z)$, it follows that:

$$\sum_{n=0}^{\infty} a_{2^t}(n) q^{8n+\left(\frac{4^t-1}{3}\right)} \equiv \Delta^{\frac{4^t-1}{3}}(z) \pmod{2}. \tag{2}$$

Let $\sum_{n=0}^{\infty} A_{2^t}(n)q^n$ be the Fourier expansion of $\Delta^{\frac{4^t-1}{3}}(z)$. We observe that $\Delta^{\frac{4^t-1}{3}}(z) \in S_{4^{t+1}-4}$. Therefore, to prove our results about the parity of 2^t-core partition functions in certain arithmetic progressions, we investigate the modular forms $\Delta^{\frac{4^t-1}{3}}(z)$ and the spaces $S_{4^{t+1}-4} \pmod{2}$. In particular, we use two properties of $S_{4^{t+1}-4} \pmod{2}$ to obtain congruences for $A_{2^t}(n) \pmod{2}$. These properties are contained in the following theorems.

Theorem 2.1. *If j is a positive integer, then $\{\Delta^r(z)\}_{r=1}^{j}$ is a basis for $S_{12j} \pmod 2$.*

Proof. It is well known that $\{E_{12j-12r}(z)\Delta^r(z)\}_{r=1}^{j}$ is a basis for S_{12j}, where, for a positive even integer m, $E_m(z)$ is the Eisenstein series of weight m. Furthermore, for any such m, it is also known that $E_m(z) \equiv 1 \pmod 2$. The result follows. □

Next, we analyze the action of the Hecke operators T_p on $S_k \pmod 2$. The Hecke operators are linear transformations on S_k defined as follows: If p is a prime, and

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_k$$

then

$$f(z) \mid T_p = \sum_{n=0}^{\infty} \left(a(pn) + p^{k-1}a\left(\frac{n}{p}\right)\right) q^n \in S_k,$$

where $a(\frac{n}{p}) = 0$ if $\frac{n}{p}$ is not an integer. Note that if p is an odd prime,

$$f(z) \mid T_p \equiv \sum_{n=0}^{\infty} \left(a(pn) + a\left(\frac{n}{p}\right)\right) q^n \pmod 2. \tag{3}$$

From [5], the action of the Hecke operators on $S_j \pmod 2$ has the following useful property. It is the primary tool used to prove Theorems 1.1 and 1.2.

Theorem 2.2. *If j is a positive integer, then there exists $h(j)$, a positive integer, such that for any collection of $h(j)$ odd primes, $p_1, \ldots, p_{h(j)}$, and for any $f(z) \in S_j$, the following identity holds:*

$$f(z) \mid T_{p_1} \cdots \mid T_{p_{h(j)}} \equiv 0 \pmod 2.$$

By computing an upper bound on the number $h(12j)$, where $j = \frac{4^t-1}{3}$, we can apply Theorem 2.2 to the modular form $\Delta^{\frac{4^t-1}{3}}(z)$, and thereby obtain congruences $\pmod 2$ for its coefficients, $A_{2^t}(n)$. We obtain such an upper bound in the following two corollaries.

Corollary 2.3. *If j is a positive integer, and p is an odd prime, then*

$$\Delta^j(z) \mid T_p \equiv \sum_{i=1}^{j} a_i \Delta^i(z) \pmod 2, \tag{4}$$

where $a_j \equiv 0 \pmod 2$.

Proof. By Theorem 2.1, we have that $\Delta^j(z) \mid T_p \equiv \sum_{i=1}^{j} a_i \Delta^i(z) \pmod 2$. We now suppose that $a_j \equiv 1 \pmod 2$. Letting $\Delta^j(z) \mid (T_p)^{(l)}$ denote the lth iteration of T_p applied to $\Delta^j(z)$, we find, by induction, that for any positive integer l, $\Delta^j(z) \mid (T_p)^{(l)} \equiv \sum_{i=1}^{j} b_i \Delta^i(z) \pmod 2$, where $b_j \equiv 1 \pmod 2$. That is, in this situation, $\Delta^j(z) \mid (T_p)^{(l)}$ is not equivalent to $0 \pmod 2$ for every l. But by Theorem 2.2, there exists $h(12j)$, a positive integer such that $\Delta^j(z) \mid (T_p)^{(h(12j))} \equiv 0 \pmod 2$, a contradiction.

□

Our upper bound on the number $h(12j)$ now follows.

Corollary 2.4. *An upper bound on the number of Hecke operators T_p, where p is an odd prime, needed to annhilate an arbitrary $f(z) \in S_{12j} \pmod 2$ is j. (That is, $h(12j) \leq j$).*

Proof. An arbitrary $f(z) \in S_{12j} \pmod 2$ is a linear combination of basis elements from $\{\Delta^r(z)\}_{r=1}^{j} \pmod 2$. By Corollary 2.3, if p_1 is an odd prime, $f(z) \mid T_{p_1}$ is a linear combination of elements from $\{\Delta^r(z)\}_{r=1}^{j-1} \pmod 2$. Proceeding inductively, we find that for any j odd primes, $f(z) \mid T_{p_1} \cdots \mid T_{p_j} \equiv 0 \pmod 2$, showing that $h(12j) \leq j$.

□

We now prove Theorems 1.1 and 1.2. Observing that $\Delta^{\frac{4^t-1}{3}}(z) \in S_{4^{t+1}-4}$, it follows by Corollary 2.4 that for any $p_1, ..., p_{\frac{4^t-1}{3}}$, odd primes, we have

$$\left(\sum_{n=0}^{\infty} A_{2^t}(n) q^n\right) \mid T_{p_1} \cdots \mid T_{p_{\frac{4^t-1}{3}}} \equiv \sum_{n=0}^{\infty} B_{2^t}(n) q^n \equiv 0 \pmod 2. \tag{5}$$

Hence, $B_{2^t}(n) \equiv 0 \pmod 2$. By placing constraints on the collection of primes and the number n, we obtain simple formulas for the coefficients $B_{2^t}(n)$ in terms of the coefficients $A_{2^t}(n)$, which we translate into the congruences for the 2^t-core partition functions stated by Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Letting $p_1, ..., p_{\frac{4^t-1}{3}}$ be distinct odd primes, if $\gcd(p_1 \cdots p_{\frac{4^t-1}{3}}, n) = 1$, the action of T_{p_1} yields nth coefficient

$$A_{2^t}(p_1 n) + A_{2^t}\left(\frac{n}{p_1}\right) = A_{2^t}(p_1 n)$$

since $p_1 \nmid n$. Proceeding inductively, we find that the nth coefficient of $\left(\sum_{n=0}^{\infty} A_{2^t}(n) q^n\right) \mid T_{p_1} \cdots \mid T_{p_{\frac{4^t-1}{3}}}$ is

$$B_{2^t}(n) \equiv A_{2^t}(p_1 \cdots p_{\frac{4^t-1}{3}} n) \equiv 0 \pmod 2$$

We now recall that the coefficients A_{2^t} and a_{2^t} are related:

$$\sum_{n=0}^{\infty} A_{2^t}(n)q^n = \Delta^{\frac{4^t-1}{3}}(z) \equiv \sum_{n=0}^{\infty} a_{2^t}(n)q^{8n+\left(\frac{4^t-1}{3}\right)} \pmod 2.$$

From this, we deduce the theorem.

□

Proof of Theorem 1.2. We prove the case of $p \equiv 3 \pmod 8$. The other cases have similar proofs. It is well known that

$$\Delta(z) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod 2. \tag{6}$$

From this, it follows that if $\Delta^j(z) \equiv \sum_{n=0}^{\infty} a_j(n)q^n \pmod 2$, then $a_j(n) \equiv 0 \pmod 2$ unless $n \equiv j \pmod 8$, in which case $a_j(n) \equiv 0$ or $1 \pmod 2$. Letting $\frac{4^t-1}{3} = 8k+5$, we bound the number of iterations of T_p needed to annhilate $\Delta^{8k+5}(z) \equiv \sum_{n=0}^{\infty} A_{2^t}(n)q^n \pmod 2$. If $\Delta^{8k+5}(z) \mid (T_p)^{(l)} \equiv \sum_{n=0}^{\infty} b_l(n)q^n \pmod 2$, by the above argument, we find that $b_1(n) \equiv A_{2^t}(pn) + A_{2^t}(\frac{n}{p}) \equiv 0 \pmod 2$, unless $n \equiv 7 \pmod 8$, in which case $b_1 \equiv 0$ or $1 \pmod 2$. Hence, by Corollary 2.3, $\Delta^{8k+5}(z) \mid T_p$ is congruent $\pmod 2$ to a linear combination of powers of $\Delta(z)$, where each power is congruent to $7 \pmod 8$, the largest of which is at most $8(k-1)+7$. Repeating this argument, we can show that $\Delta^{8k+5}(z) \mid (T_p)^{(2)}$ is congruent $\pmod 2$ to a linear combination of powers of $\Delta(z)$, each congruent to $5 \pmod 8$ and not exceeding $8(k-1)+5$. By induction, we find that $\Delta^{8k+5}(z) \mid (T_p)^{(2k+1)} \equiv 0 \pmod 2$, and hence, that

$$\Delta^{\frac{4^t-1}{3}}(z) \mid (T_p)^{\left(\frac{4^{t-1}-1}{3}\right)} \equiv 0 \pmod 2. \tag{7}$$

One can also show, using induction, that for any positive integer j, $(\sum_{n=0}^{\infty} A_{2^t}(n)q^n) \mid (T_p)^{(2^j-1)} \pmod 2$ has nth coefficient $A_{2^t}(p^{2^j-1}n)$ if $\gcd(n,p) = 1$. In this case, if j is a positive integer and $j \geq 2t-3$, then $2^j - 1 \geq \frac{4^{t-1}-1}{3}$, so that $(\sum_{n=0}^{\infty} A_{2^t}(n)q^n) \mid (T_p)^{(2^{2t-3}-1)} \equiv 0 \pmod 2$. Hence, $A_{2^t}(p^{2^{2t-3}}n) \equiv 0 \pmod 2$. As in Theorem 1.1, we translate this into a statement about 2^t-core partition functions to prove the theorem.

□

3. Specific examples

To prove our general congruences, we derived upper bounds for the number of Hecke operators needed to annhilate $\Delta^{\frac{4^t-1}{3}}(z) \pmod 2$. These upper bounds are not always sharp. In our first two examples, we exhibit collections of significantly fewer distinct odd primes than $\frac{4^t-1}{3}$, the upper bound used in Theorem 1.1.

Example 3.1: If $t=3$, the upper bound used in Theorem 1.1 is 21. However, by choosing primes $3,5,7$, we find that $(\sum_{n=0}^{\infty} A_8(n)q^n) \mid T_3 \mid T_5 \mid T_7 \equiv 0 \pmod 2$. Furthermore, if $\gcd(105,n)=1$, we have:

$$a_8(105n+63) \equiv 0 \pmod 2. \tag{8}$$

Example 3.2: If $t=4$, the upper bound used in Theorem 1.1 is 85, but we find three primes, by choosing $3,5,17$, such that $(\sum_{n=0}^{\infty} A_{16}(n)q^n) \mid T_3 \mid T_5 \mid T_{17} \equiv 0 \pmod 2$. If $\gcd(255,n)=1$, we have:

$$a_{16}(255n+85) \equiv 0 \pmod 2. \tag{9}$$

In examples 3.3-3.5, we consider the iterated action of a single Hecke operator associated to a prime, p. We compare our calculated examples to the congruences given by Theorem 1.2.

Example 3.3: If $p=5$, we find that $(\sum_{n=0}^{\infty} A_8(n)q^n) \mid (T_5)^{(4)} \equiv 0 \pmod 2$. Since the coefficients of the resulting modular form contain multiple summands, we use that the nth coefficient of $(\sum_{n=0}^{\infty} A_8(n)q^n) \mid (T_5)^{(7)}$ is $A_8(5^7n)$ if $\gcd(5,n)=1$. Then for every positive integer n, and for every $k \in \{0,...,4\}, k \neq 3$:

$$a_8\left(\frac{5^7(40n+8k+1)-21}{8}\right) \equiv 0 \pmod 2, \tag{10}$$

which coincides with the congruence obtained by applying Theorem 1.2.

Example 3.4: If $p=7$, we have that $(\sum_{n=0}^{\infty} A_8(n)q^n) \mid (T_7)^{(2)} \equiv 0 \pmod 2$, and $(\sum_{n=0}^{\infty} A_{16}(n)q^n) \mid (T_7)^{(4)} \equiv 0 \pmod 2$. Reasoning as in the previous example, we use that the nth coefficient of $(\sum_{n=0}^{\infty} A_8(n)q^n) \mid (T_7)^{(3)}$ is $A_8(7^3n)$, and of $(\sum_{n=0}^{\infty} A_{16}(n)q^n) \mid (T_7)^{(7)}$, is $A_{16}(7^7n)$. Hence, for every positive integer n, and for every $k \in \{0,...,6\}, k \neq 4$ it follows that:

$$a_8\left(\frac{7^3(56n+8k+3)-21}{8}\right) \equiv 0 \pmod 2, \tag{11}$$

and

$$a_{16}\left(\frac{7^7(56n+8k+3)-85}{8}\right) \equiv 0 \pmod 2, \tag{12}$$

whereas the powers of 7 in the moduli of the congruences obtained by applying Theorem 1.2 in these cases are 7 and 31 respectively.

Example 3.5: If $p = 11$, we see that $\left(\sum_{n=0}^{\infty} A_8(n)q^n\right) \mid (T_{11})^{(3)} \equiv 0 \pmod 2$ and so, for $k \in \{0, ..., 10\}, k \neq 6$:

$$a_8\left(\frac{11^3(88n+8k+7)-21}{8}\right) \equiv 0 \pmod 2. \tag{13}$$

The power of 11 in the modulus of the congruence obtained by applying Theorem 1.2 in this case is 7.

References

1. F. Garvan, D. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), 1-17.
2. M. Hirschhorn and J.A. Sellers, *Some parity results for* 16-*cores*, Ramanujan Journal **3** (1999), 281-296.
3. M. Hirschhorn and J.A. Sellers, *Some amazing facts about* 4-*cores*, Journal of Number Theory **60** (1996), 51-69.
4. L. W. Kolitsch and J.A. Sellers, *Elementary proofs of infinitely many congruences for* 8-*cores*, Ramanujan Journal **3** (1999), 221-226.
5. J.-P. Serre, *Divisibilité de certaines fonctions arithmétique*, L'Ens. Math. **22** (1976), 227-260.

Department of Mathematics, University of Wisconsin, Madison, WI, 53706
E-mail address: boylan@math.wisc.edu