AMS Proceedings Style
Volume **00**, XXXX

Swinnerton-Dyer Type Congruences for certain Eisenstein series

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Jan 11, 2000.

Abstract. We consider a normalized Eisenstein series of weight k on a congruence subgroup of type $\Gamma_0(N)$ with Nebentypus character χ which vanishes at all cusps of $\Gamma_0(N)$ inequivalent to the cusp at infinity. We determine conditions on N, k, χ, and an ideal $\mathfrak{a}$ in certain number fields, under which their Fourier series are congruent to $1 \pmod{\mathfrak{a}}$.

1. Introduction and Statement of Results

If $k \geq 4$ is an even integer, then it is well-known [**K, pg.111**] that the normalized Eisenstein series given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

is a modular form of weight k with respect to $SL(2, \mathbb{Z})$, where B_k is the kth Bernoulli number, $q := e^{2\pi i z}$, and $\sigma_k(n)$ is the function which sums the kth powers of the positive divisors of n. Swinnerton-Dyer [**Sw-D**] showed that $E_k(z)$ satisfies the following congruence property:

Theorem (Swinnerton-Dyer). *If $\ell \geq 5$ is a prime, then $E_k(z) \equiv 1 \pmod{\ell}$ if and only if $k \equiv 0 \pmod{\ell - 1}$.*

This follows from the Von Staudt-Claussen Theorem regarding the divisibility of the denominators of Bernoulli numbers.

Here we generalize Swinnerton-Dyer's result to certain Eisenstein series in spaces of modular forms of weight k on a congruence subgroup of type $\Gamma_0(N)$ with Nebentypus character χ. These spaces are denoted by $M_k(\Gamma_0(N), \chi)$. For background on integer weight modular forms, see [**K**]. Following the methods in Sections 1-3 of Chapter VII of Schoeneberg's book, *Elliptic Modular Functions*, we develop the Fourier expansion at infinity of a normalized Eisenstein series $E_{N,k,\chi}(z) \in M_k(\Gamma_0(N), \chi)$ which vanishes at all cusps of $\Gamma_0(N)$ inequivalent to

1991 *Mathematics Subject Classification.* Primary 11F33.
Key words and phrases. Congruences for Eisenstein series.

the cusp at infinity, and we state conditions on N, k, and χ guaranteeing the existence of these series. (Schoeneberg does this for Eisenstein series without character on an arbitrary subgroup of level N). Using these expansions, we obtain conditions on N, k, and χ, and an ideal $\mathfrak{a}$ in certain number fields, under which $E_{N,k,\chi}(z) \equiv 1 \pmod{\mathfrak{a}}$.

Theorem 1.1 lists formulas for $E_{N,k,\chi}(z)$ when they exist.

THEOREM 1.1. *Suppose χ is a Dirichlet character with modulus N and conductor f, and $\tau_m(d,\chi) := \sum_{h=1}^{m-1} \chi(h)\zeta_m^{dh}$, where $\zeta_m := e^{\frac{2\pi i}{m}}$. Suppose also that if χ is nontrivial and $N = 1$ or 2, then $k \geq 4$ is an even integer satisfying $\chi(-1) = (-1)^k$, and if χ is nontrivial and $N > 2$, then $k \geq 3$ is an integer satisfying $\chi(-1) = (-1)^k$. Then the series $E_{N,k,\chi}(z)$ given by the following formulas are normalized modular forms in $M_k(\Gamma_0(N),\chi)$ which vanish at all cusps of $\Gamma_0(N)$ inequivalent to the cusp at infinity.*

1. *If χ is trivial and $N = 1$, then for an even integer $k \geq 4$,*

$$E_{N,k,\chi}(z) = E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n. \tag{1}$$

If χ is trivial and $N > 1$, then for an even integer $k \geq 4$,

$$E_{N,k,\chi}(z) = 1 - \frac{2k\phi(N)}{N^k B_k \prod_{p|N}\left(1 - \frac{1}{p^k}\right)} \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ d>0}} d^{k-1} \frac{\mu(N/\gcd(d,N))}{\phi(N/\gcd(d,N))} \right) q^n, \tag{2}$$

where ϕ denotes Euler's phi function, and μ denotes the Möbius function.

2. *If χ is nontrivial, then*

$$E_{N,k,\chi}(z) = 1 - \frac{k}{(\frac{N}{f})^k \tau_f(1,\overline{\chi}) B_{k,\chi}} \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ d>0}} d^{k-1} (\tau_N(d,\overline{\chi}) + (-1)^k \tau_N(-d,\overline{\chi})) \right) q^n, \tag{3}$$

where $B_{k,\chi}$ is the generalized Bernoulli number associated to χ.

3. *If χ is nontrivial and primitive, then*

$$E_{N,k,\chi}(z) = 1 - \frac{2k}{B_{k,\chi}} \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ d>0}} \chi(d) d^{k-1} \right) q^n. \tag{4}$$

In what follows, let $K_\chi = \mathbb{Q}(\chi)$ denote the extension of $\mathbb{Q}$ obtained by adjoining the values of χ, let O_{K_χ} denote the ring of integers of K_χ, and denote by $O_{K_{\chi,N}}$ the ring of integers of $K_{\chi,N} = \mathbb{Q}(\chi,\zeta_N)$. We also define $\mathrm{ord}_m(n)$ to be the power of m dividing n if m and n are integers. If $\alpha = \frac{a}{b} \in \mathbb{Q}$, then $\mathrm{ord}_m(\alpha) := \mathrm{ord}_m(a) - \mathrm{ord}_m(b)$. Theorems 1.2 and 1.3 generalize Swinnerton-Dyer's Theorem to the series (2) and (4) listed in Theorem 1.1.

THEOREM 1.2. *Suppose that ℓ is an odd prime, χ is the trivial Dirichlet character modulo N, and $k \geq 4$ is an even integer. Then the following are true:*

1. $E_{N,k,\chi}(z) \equiv 1 \pmod{\ell}$ *if and only if* $k \equiv 0 \pmod{\ell - 1}$ *and* $N = \ell^t$ *for some nonnegative integer t.*
2. $E_{N,k,\chi}(z) \equiv 1 \pmod{2}$ *if and only if* $N = 2^a p^b$, *where p is an odd prime satisfying* $\mathrm{ord}_2(p^k - 1) = 1 + \mathrm{ord}_2(k)$ *and a and b are nonnegative integers.*

REMARK. *Theorems 1.2.1 and 1.2.2 contain Swinnerton-Dyer's Theorem as a special case, the case where $N = 1$.*

THEOREM 1.3. *Suppose that ℓ is an odd rational prime, $\mathfrak{a}$ is an ideal in $O_{K_\chi,N}$ with the property that $\mathfrak{a} \nmid (2)$, and χ is a nontrivial primitive Dirichlet character. Then for an integer $k \geq 3$ satisfying $\chi(-1) = (-1)^k$, we have:*

1. *If N has at least two distinct prime divisors, then*
$$E_{N,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}.$$
2. *If $N = 4$ and $\mathfrak{a} \nmid (4)$, then*
$$E_{4,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}.$$
If $N = 2^t$ for some integer $t \geq 3$, then
$$E_{2^t,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}.$$
3. *If $N = \ell$, then*
$$E_{\ell,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$$
unless there is a primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$ satisfying
$$\mathfrak{p} = \gcd(\ell, 1 - \chi(g)g^k) \neq (1),$$
where $\mathfrak{p}$ is an ideal in O_{K_χ}. In this case
$$E_{\ell,k,\chi}(z) \equiv 1 \pmod{\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\ell)}}.$$
4. *If $N = \ell$ and $\chi = \left(\frac{\bullet}{\ell}\right)$, the Legendre symbol, then*
$$E_{\ell,k,\left(\frac{\bullet}{\ell}\right)} \equiv 1 \pmod{\ell}$$
if and only if $k \equiv \frac{\ell-1}{2} \pmod{\ell - 1}$.
5. *If $N = \ell^t$ for some integer $t \geq 2$, and if $\gcd(\ell, 1 - \chi(g)g^k) = (1)$ for every primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$, then*
$$E_{\ell^t,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}.$$

Note that if $\mathfrak{a}$ is an ideal in $O_{K_\chi,N}$ and if j is a positive integer, then $E^j_{N,k,\chi}(z) \equiv 1 \pmod{\mathfrak{a}}$ whenever $E_{N,k,\chi}(z) \equiv 1 \pmod{\mathfrak{a}}$, where $E^j_{N,k,\chi}(z) \in M_{jk}(\Gamma_0(N), \chi)$.

2. Background on Schoeneberg's Eisenstein Series

Before proceeding with the proof of Theorems 1.1-1.3, we describe the basic properties of Schoeneberg's primitive and reduced Eisenstein series, the building blocks for the Eisenstein series $E_{N,k,\chi}(z)$ that we construct. We keep the notation from Schoeneberg's book in what follows.

If $f : \mathbb{H} \mapsto \hat{\mathbb{C}}$, where $\mathbb{H}$ is the upper half plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and if $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$, then $f(z)|_k S := (cz+d)^{-k} f(Sz)$. Suppose that $N \geq 1$ and $k \geq 3$ are integers, and $\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ are pairs of integers. Schoeneberg defines the inhomogenous Eisenstein series of weight k and level N as follows [**Sc, pg.155, (2)**]:

$$G_{N,k,\mathbf{a}}(z) = \sum_{\substack{m_1 \equiv a_1 \pmod N \\ m_2 \equiv a_2 \pmod N \\ \mathbf{m} \neq \mathbf{0}}} (m_1 z + m_2)^{-k},$$

(In his notation, Schoeneberg refers to modular forms having *dimension* $-k < 0$, rather than having *weight* $k > 0$, which means the same. We prefer to use the term *weight*.) If $\gcd(a_1, a_2, N) = 1$, then $G_{N,k,\mathbf{a}}(z)$ is called a *primitive* Eisenstein series.

The relevant facts about primitive Eisenstein series are these:

1. [**Sc, pg.155, Thm.1**] For all $\mathbf{a}$, $G_{N,k,\mathbf{a}}(z) \in M_k(\Gamma(N))$.
2. [**Sc, pg.155, (3)**] For all $\mathbf{a}$,

$$G_{N,k,-\mathbf{a}}(z) = (-1)^k G_{N,k,\mathbf{a}}(z). \tag{5}$$

3. [**Sc, pg.155, (3)**] If $\mathbf{a} \equiv \mathbf{a}_1 \pmod N$, then

$$G_{N,k,\mathbf{a}}(z) = G_{N,k,\mathbf{a}_1}(z). \tag{6}$$

4. [**Sc, pg.156, (4)**] If $A \in SL(2,\mathbb{Z})$, then

$$G_{N,k,\mathbf{a}}(z)|_k A = G_{N,k,A'\mathbf{a}}(z), \tag{7}$$

   where A' is the transpose of A.
5. [**Sc, pg.157, (5)**] For integers a and b, define

$$\delta\left(\frac{a}{b}\right) = \begin{cases} 1 & \text{if } b \mid a, \\ 0 & \text{if } b \nmid a. \end{cases} \tag{8}$$

   Then

$$G_{N,k,\mathbf{a}}(z) = \sum_{\nu \geq 0} \alpha_\nu(N,k,\mathbf{a}) e^{\frac{2\pi i \nu z}{N}},$$

   where

$$\alpha_0(N,k,\mathbf{a}) = \delta\left(\frac{a_1}{N}\right) \sum_{\substack{m_2 \equiv a_2 \pmod N \\ m_2 \neq 0}} m_2^{-k}, \tag{9}$$

and for $\nu \geq 1$,

$$\alpha_\nu(N,k,\mathbf{a}) = \frac{(-2\pi \mathrm{i})^k}{N^k(k-1)!} \sum_{\substack{m|\nu \\ \frac{\nu}{m}\equiv a_1 \pmod N}} m^{k-1}\mathrm{sgn}(m)\zeta_N^{a_2 m}. \tag{10}$$

One may also define *reduced* Eisenstein series. If $\gcd(a_1,a_2,N)=1$, then these may be written [**Sc, pg.158, (7)**]:

$$G^*_{N,k,\mathbf{a}}(z) = \sum_{\substack{\mathbf{m}\equiv\mathbf{a} \pmod N \\ \gcd(m_1,m_2)=1}} (m_1 z + m_2)^{-k}.$$

Facts 1-4 concerning primitive Eisenstein series also hold for reduced Eisenstein series. The reduced Eisenstein series are expressible as a linear combination of primitive Eisenstein series [**Sc, pg.159, (9)**]:

$$G^*_{N,k,\mathbf{a}}(z) = \sum_{t \pmod N} \left(\sum_{\substack{dt\equiv 1 \pmod N \\ d>0}} \frac{\mu(d)}{d^k} \right) G_{N,k,t\mathbf{a}}(z). \tag{11}$$

We now proceed with the proof of Theorem 1.1

3. The Proof of Theorem 1.1

We note that $\Gamma_0(N) = \bigcup_{\nu=1}^{\mu_1} \Gamma(N)A_\nu$, where the coset representatives A_ν lie in the set

$$\left\{ \begin{bmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{bmatrix} \in SL(2,\mathbb{Z}) \right\}, \tag{12}$$

where $\alpha_\nu \in (\mathbb{Z}/N\mathbb{Z})^*$, $\beta_\nu \in \mathbb{Z}/N\mathbb{Z}$, $\gamma_\nu \equiv 0 \pmod N$, $\delta_\nu \equiv \alpha_\nu^{-1} \pmod N$, and $\mu_1 = [\Gamma_0(N) : \Gamma(N)]$. We suppose that χ is a Dirichlet character with modulus N and conductor f. As our goal is to construct modular forms for $M_k(\Gamma_0(N),\chi)$, we impose the condition that $k \geq 3$ is an integer with the property that

$$\chi(-1) = (-1)^k. \tag{13}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, then the transformation law satisfied by modular forms $f(z) \in M_k(\Gamma_0(N),\chi)$ is given by:

$$f(z)|_k A = \chi(d) f(z). \tag{14}$$

An application of (14) using $A = -I \in \Gamma_0(N)$ shows that the spaces $M_k(\Gamma_0(N),\chi)$ contain only the modular form which is identically zero when (13) does not hold. We claim that

$$G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) = \sum_{\nu=1}^{\mu_1} \overline{\chi}(\delta_\nu) G^*_{N,k,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A_\nu$$

is a modular form in $M_k(\Gamma_0(N),\chi)$ with the property that it vanishes at all cusps of $\Gamma_0(N)$ inequivalent to the cusp at infinity.

Observing that $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ is a linear combination of reduced, and hence, by (11), primitive Eisenstein series, we impose the additional condition that $k \geq 4$ is an even integer when $N = 1$ or 2. When $N = 1$ or 2 and $k \geq 3$ is an odd integer, it follows by (6) and (5) that

$$\begin{aligned} G_{N,k,\mathbf{a}}(z) &= G_{N,k,-\mathbf{a}}(z) \\ &= -G_{N,k,\mathbf{a}}(z). \end{aligned}$$

This shows that $G_{N,k,\mathbf{a}}(z) = 0$ in this case.

To verify that $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) \in M_k(\Gamma_0(N),\chi)$, we only need to show that it satisfies (14) since it clearly satisfies the remaining defining properties of a modular form in $M_k(\Gamma_0(N),\chi)$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, then $A_\nu A = G_\nu A_{\nu'}$ for some $G_\nu \in \Gamma(N)$, and for some ν' uniquely determined by ν which runs through $\{1, ..., \mu_1\}$ as ν does. Moreover, $A_{\nu'} = \begin{bmatrix} \alpha_{\nu'} & \beta_{\nu'} \\ \gamma_{\nu'} & \delta_{\nu'} \end{bmatrix}$, with $\delta_\nu \equiv d^{-1}\delta_{\nu'} \pmod N$. Therefore,

$$\begin{aligned} G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A &= \sum_{\nu=1}^{\mu_1} \overline{\chi}(\delta_\nu) G^*_{N,k,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A_\nu A \\ &= \sum_{\nu'=1}^{\mu_1} \overline{\chi}(d^{-1}\delta_{\nu'}) G^*_{N,k,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k G_\nu A_{\nu'} \\ &= \chi(d) \sum_{\nu'=1}^{\mu_1} \overline{\chi}(\delta_{\nu'}) G^*_{N,k,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A_{\nu'} \\ &= \chi(d) G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z), \end{aligned}$$

so $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) \in M_k(\Gamma_0(N),\chi)$.

Next, we calculate the value of $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ at an arbitrary cusp $\frac{-d}{c}$. To do this, we form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$, and consider $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A^{-1} = \sum_{n\geq 0} r(n) q^{\frac{n}{N}}$. The value of $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ at $\frac{-d}{c}$ is $r(0)$. The first step in the calculation is to simplify $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A^{-1}$ using (7) twice and (12):

$$\begin{aligned} G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A^{-1} &= \sum_{\nu=1}^{\mu_1} \overline{\chi}(\delta_\nu) G^*_{N,k,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)|_k A_\nu A^{-1} \\ &= \sum_{\nu=1}^{\mu_1} \overline{\chi}(\delta_\nu) G^*_{N,k,\left[\begin{smallmatrix}\gamma_\nu\\\delta_\nu\end{smallmatrix}\right]}(z)|_k A^{-1} \\ &= N \sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(h) G^*_{N,k,\left[\begin{smallmatrix}0\\h\end{smallmatrix}\right]}(z)|_k A^{-1} \\ &= N \sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(h) G^*_{N,k,\left[\begin{smallmatrix}-ch\\ah\end{smallmatrix}\right]}(z). \end{aligned} \tag{15}$$

In view of (15), we simplify $G^*_{N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]}(z)$ using (11):

$$G^*_{N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]}(z) = \sum_{\nu\geq 0} \alpha^*_\nu\left(N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]\right) e^{\frac{2\pi i\nu z}{N}}$$

$$= \sum_{t\in(\mathbb{Z}/N\mathbb{Z})^*} \left(\sum_{\substack{dt\equiv 1 \pmod N\\ d>0}} \frac{\mu(d)}{d^k} \right) G_{N,k,\left[\begin{smallmatrix}-tch\\ tah\end{smallmatrix}\right]}(z). \tag{16}$$

We follow Schoeneberg's computation of $\alpha^*_0\left(N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]\right)$ and note that the first equality is obtained by applying (9) and (16) [**Sc, pg.160**]:

$$\alpha^*_0\left(N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]\right) = \sum_{t\in(\mathbb{Z}/N\mathbb{Z})^*} \left(\sum_{\substack{dt\equiv 1 \pmod N\\ d>0}} \frac{\mu(d)}{d^k} \right) \delta\left(\frac{-tch}{N}\right) \sum_{\substack{m\equiv tah \pmod N\\ m\neq 0}} m^{-k}$$

$$= \delta\left(\frac{-ch}{N}\right) \sum_{d>0} \sum_{\substack{md\equiv ah \pmod N\\ m\neq 0}} \frac{\mu(d)}{(dm)^k}$$

$$= \delta\left(\frac{-ch}{N}\right) \sum_{\substack{m\equiv ah \pmod N\\ m\neq 0}} m^{-k} \sum_{\substack{d\mid m\\ d>0}} \mu(d) \tag{17}$$

Observe that since

$$\sum_{\substack{d\mid m\\ d>0}} \mu(d) = \begin{cases} 1 & \text{if } m = 1, \text{or} -1, \\ 0 & \text{if } m \neq 1, \text{or} -1, \end{cases} \tag{18}$$

it follows that $\alpha^*_0\left(N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]\right) \neq 0$ if and only if $c \equiv 0 \pmod N$ and $d \equiv h \pmod N$, i.e., if and only if $\frac{-d}{c}$ is $\Gamma(N)$-equivalent to the cusp at infinity. Continuing our calculation, we now have

$$r(0) = N \sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(h)\alpha^*_0\left(N,k,\left[\begin{smallmatrix}-ch\\ ah\end{smallmatrix}\right]\right) \tag{19}$$

by (15). Therefore, $r(0) = 0$ at all cusps of $\Gamma(N)$ inequivalent to the cusp at infinity, and hence, at all cusps of $\Gamma_0(N)$ inequivalent to the cusp at infinity since $\Gamma(N) \subset \Gamma_0(N)$.

It remains to show that the value of $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]}(z)$ at the cusp at infinity is nonzero. Combining the previous facts given by (17), (19), (18), and (13), we calculate:

$$r(0) = N \sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(h) \sum_{\substack{m\equiv h \pmod N\\ m\neq 0}} m^{-k} \sum_{\substack{d\mid m\\ d>0}} \mu(d)$$

$$= N(\overline{\chi}(1)(1)^k + \overline{\chi}(-1)(-1)^k)$$

$$= 2N. \tag{20}$$

This proves that $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ satisfies the cusp conditions stated in Theorem 1.1.

We now develop the normalized Fourier expansion of $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ at infinity. Recall that k is even when $N=1$ or 2. Letting $s_t := \sum_{\substack{dt\equiv 1 \pmod N\\ d>0}} \frac{\mu(d)}{d^k}$ for $t\in(\mathbb{Z}/N\mathbb{Z})^*$, we simplify $G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z)$ using (15) and (11):

$$\begin{aligned}
G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) &= N\sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(h)G^*_{N,k,\left[\begin{smallmatrix}0\\h\end{smallmatrix}\right]}(z)\\
&= N\sum_{h\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(h)\sum_{t\in(\mathbb{Z}/N\mathbb{Z})^*}s_tG_{N,k,\left[\begin{smallmatrix}0\\th\end{smallmatrix}\right]}(z).
\end{aligned}$$

Letting $c(N) := N\sum_{t\in(\mathbb{Z}/N\mathbb{Z})^*}s_t\chi(t)$, a constant dependent only on N, and making the change of variable $j=th$, we obtain

$$G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) = c(N)\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)G_{N,k,\left[\begin{smallmatrix}0\\j\end{smallmatrix}\right]}(z). \tag{21}$$

Substituting (9) and (10) in (21), we have

$$\begin{aligned}
&G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) = c(N)\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\times\\
&\left(\sum_{\substack{m\equiv j \pmod N\\ m\neq 0}} m^{-k} + \frac{(-2\pi \mathrm{i})^k}{N^k(k-1)!}\sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 0 \pmod N}} d^{k-1}\operatorname{sgn}(d)\zeta_N^{dj}\right)q^{\frac{n}{N}}\right).
\end{aligned} \tag{22}$$

Using (22) and (13), observe that the constant term $r(0)$ may also be expressed as

$$\begin{aligned}
r(0) &= c(N)\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\sum_{\substack{m\equiv j \pmod N\\ m\neq 0}} m^{-k}\\
&= c(N)\sum_{m\geq 1}\overline{\chi}(m)m^{-k}(1+\overline{\chi}(-1)(-1)^k)\\
&= 2c(N)L(k,\overline{\chi}),
\end{aligned} \tag{23}$$

where $L(k,\chi)$ is the Dirichlet L-function associated to χ. Substituting (23) in (22) gives us

$$G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) = c(N)\times$$
$$\left(2L(k,\overline{\chi}) + \frac{(-2\pi \mathrm{i})^k}{N^k(k-1)!}\sum_{n\geq 1}\left(\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 0 \pmod N}} d^{k-1}sgn(d)\zeta_N^{dj}\right)q^{\frac{n}{N}}\right).$$

It is clear from (20) and (23) that $c(N)\neq 0$. Therefore, we can define

$$E_{N,k,\chi}(z) := (2L(k,\overline{\chi})c(N))^{-1}G^*_{\Gamma_0(N),k,\chi,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(z) = 1 + \frac{(-2\pi \mathrm{i})^k}{2N^k(k-1)!L(k,\overline{\chi})}\times$$
$$\sum_{n\geq 1}\left(\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 0 \pmod N}} d^{k-1}\mathrm{sgn}(d)\zeta_N^{dj}\right)q^{\frac{n}{N}} \in M_k(\Gamma_0(N),\chi).$$

Noting that

$$\begin{aligned}\sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\sum_{d\mid n} d^{k-1}\mathrm{sgn}(d)\zeta_N^{dj} &= \sum_{j\in(\mathbb{Z}/N\mathbb{Z})^*}\overline{\chi}(j)\sum_{\substack{d\mid n\\ d>0}} d^{k-1}(\zeta_N^{dj}+(-1)^k\zeta_N^{-dj})\\ &= \sum_{\substack{d\mid n\\ d>0}} d^{k-1}(\tau_N(d,\overline{\chi})+(-1)^k\tau_N(-d,\overline{\chi})),\end{aligned}$$

and that the condition $\frac{n}{d}\equiv 0 \pmod N$ allows us to make the change of variable $n\to nN$, our formula becomes:

$$E_{N,k,\chi}(z) = 1 + \frac{(-2\pi \mathrm{i})^k}{2N^k(k-1)!L(k,\overline{\chi})}\sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ d>0}} d^{k-1}(\tau_N(d,\overline{\chi})+(-1)^k\tau_N(-d,\overline{\chi}))\right)q^n.$$

We now simplify $E_{N,k,\chi}(z)$ in the three cases specified in Theorem 1.1 using certain well-known facts. If χ is the trivial character with modulus N, then $\overline{\chi}=\chi$ and k is even. We use the following facts to obtain formulas (1) and (2):

1. [**Ir-R, Thm.2, pg.231**] If k is a positive even integer, then

$$2\zeta(k) = \frac{(-1)^{\frac{k}{2}+1}(2\pi)^k B_k}{k!},$$

   where $\zeta(s)$ is the Riemann zeta function.
2. [**Ir-R, pg.255**] If χ is trivial, then

$$L(k,\chi) = \begin{cases}\zeta(k) & \text{if } N=1,\\ \zeta(k)\prod_{p\mid N}\left(1-\frac{1}{p^k}\right) & \text{if } N>1.\end{cases}$$

3. [**A, pg.164**] If χ is trivial, then

$$\begin{aligned}\tau_N(d,\chi) &= \tau_N(-d,\chi) \\ &= \frac{\phi(N)\mu(N/\gcd(d,N))}{\phi(N/\gcd(d,N))}.\end{aligned}$$

When χ is a nontrivial character with modulus N and conductor f, we use a different set of facts to produce the formula (3):

1. [**Ir-R, Prop.16.6.2**] If χ is nontrivial and k is a positive integer, then

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k}.$$

2. If k is a positive integer, then

$$\Gamma(k) = (k-1)!,$$

   where $\Gamma(s)$ is the classical Γ-function.
3. If χ is nontrivial and k is a positive integer, and if we define

$$\delta_\chi := \begin{cases} 1 & \text{if } \chi(-1) = -1, \\ 0 & \text{if } \chi(-1) = 1, \end{cases}$$

   then the functional equation for $L(k,\chi)$ is [**Iw, Ch.1, Sec.1.2**]:

$$L(k,\chi) = \frac{\tau_f(1,\chi)}{2\mathrm{i}^{\,\delta_\chi}} \left(\frac{2\pi}{f}\right)^k \frac{L(1-k,\overline{\chi})}{\Gamma(k)\cos\left(\frac{\pi(k-\delta_\chi)}{2}\right)}.$$

If χ is a nontrivial primitive character with modulus N, we know the additional fact [**A, Thm.8.15**]:

$$\tau_N(d,\chi) = \chi(d)\tau_N(1,\chi),$$

which we substitute in (3) to obtain (4). This finishes the proof of Theorem 1.1.

4. The Proof of Theorem 1.2

The proof of Theorem 1.2 relies on some well-known facts about the ordinary Bernoulli numbers, B_k:

THEOREM 4.1 (VON STAUDT-CLAUSSEN) [**Ir-R, Thm.3, pg.233**]. *Suppose that ℓ is a prime and k is a positive even integer. If $\ell - 1 \nmid k$, then* $\mathrm{ord}_\ell(B_k) \geq 0$, *and if $\ell - 1 \mid k$, then* $\mathrm{ord}_\ell(B_k) = -1$.

THEOREM 4.2 [**Ir-R, Thm.15.2.4**]. *Suppose that ℓ is a prime and k is a positive even integer. If $\ell - 1 \nmid k$, then* $\mathrm{ord}_\ell\left(\frac{B_k}{k}\right) \geq 0$.

In the case where χ is the trivial character modulo 1 and ℓ is a rational prime, the desired result follows by applying Theorem 4.1 to formula (1). This is Swinnerton-Dyer's Theorem.

Therefore, we consider the cases in which $N > 1$. We let $E_{N,k,\chi}(z) = \sum_{n\geq 0} a(n)q^n$ and $c_N := \frac{N}{\prod_{p|N} p}$. To start, we simplify the coefficient $a(c_N)$:

$$a(c_N) = \frac{-2k\phi(N)}{N^k B_k \prod_{p|N}\left(1 - \frac{1}{p^k}\right)} \sum_{d|\frac{N}{\prod_{p|N} p}} d^{k-1} \frac{\mu(N/\gcd(d,N))}{\phi(N/\gcd(d,N))}.$$

Since

$$\mu(N/\gcd(d,N)) = \begin{cases} (-1)^{\omega(N)} & \text{if } d = N/\prod_{p|N} p, \\ 0 & \text{if } d < N/\prod_{p|N} p, \end{cases}$$

where $\omega(N)$ is the number of distinct prime divisors of N, it follows that

$$a(c_N) = \frac{(-1)^{\omega(N)+1}2k}{B_k \prod_{p|N}(p^k - 1)}.$$

Note that

$$\mathrm{ord}_\ell(a(c_N)) = \mathrm{ord}_\ell(2) + \mathrm{ord}_\ell(k) - \mathrm{ord}_\ell(B_k) - \sum_{p|N} \mathrm{ord}_\ell(p^k - 1) \tag{24}$$

for a given prime ℓ. Assuming for now that ℓ is odd, we analyze $a(c_N) \pmod{\ell}$ in several cases.

CASE 1. $\ell - 1 \nmid k$.

$\mathrm{ord}_\ell\left(\frac{B_k}{k}\right) \geq 0$ by Theorem 4.2, so $\mathrm{ord}_\ell(a(c_N)) \leq 0$ by (24), and hence, $E_{N,k,\chi}(z) \not\equiv 1 \pmod{\ell}$.

Cases 2a and 2b concern the situation where $\ell - 1 \mid k$. In this situation, Theorem 4.1 implies that $\mathrm{ord}_\ell(B_k) = -1$. We suppose that $\mathrm{ord}_\ell(k) = j$. Then

$$k = \ell^j(\ell - 1)m = \phi(\ell^{j+1})m \tag{25}$$

for some positive integer m coprime to ℓ.

CASE 2A. $\ell - 1 \mid k$ *and* $\ell \nmid N$.

$\gcd(p, \ell) = 1$ for every prime $p \mid N$ since $\ell \nmid N$, so $p^k \equiv 1 \pmod{\ell^{j+1}}$ by (25). Using (24) we have

$$\mathrm{ord}_\ell(a(c_N)) \leq (j + 1)(1 - \omega(N)).$$

$\omega(N) \geq 1$ since $N > 1$, so $\mathrm{ord}_\ell(a(c_N)) \leq 0$ in this case. Consequently, $E_{N,k,\chi}(z) \not\equiv 1 \pmod{\ell}$.

CASE 2B. $\ell - 1 \mid k$ *and* $\ell \mid N$.

Using (24) and (25) again, we have

$$\mathrm{ord}_\ell(a(c_N)) \leq (j + 1)(2 - \omega(N)).$$

It follows that if N is not a positive power of ℓ, then $\mathrm{ord}_\ell(a(c_N)) \leq 0$, in which case $E_{N,k,\chi}(z) \not\equiv 1 \pmod{\ell}$.

Therefore, in the case where ℓ is an odd prime we know the following: $E_{N,k,\chi}(z) \not\equiv 1 \pmod{\ell}$ if $k \not\equiv 0 \pmod{\ell-1}$ or if $N \neq \ell^t$ for all positive integers t. We now prove the converse.

For an odd prime ℓ and a positive integer t, we simplify $E_{\ell^t,k,\chi}(z)$ by first observing that

$$\begin{aligned}\frac{\phi(\ell^t)\mu(\ell^t/\gcd(d,\ell^t))}{\phi(\ell^t/\gcd(d,\ell^t)} &= \begin{cases} \ell^{t-1}(\ell-1) & \text{if } \operatorname{ord}_\ell(d) = t, \\ -\ell^{t-1} & \text{if } \operatorname{ord}_\ell(d) = t-1, \\ 0 & \text{if } \operatorname{ord}_\ell(d) \leq t-2. \end{cases} \\ &= \ell^{t-1}\left(\delta\left(\frac{d}{\ell^t}\right)\ell - \delta\left(\frac{d}{\ell^{t-1}}\right)\right), \end{aligned} \tag{26}$$

using the notation defined by (8). After substituting (26) in (2), we obtain

$$\begin{aligned} E_{\ell^t,k,\chi}(z) = 1 - \frac{2k\ell^{t-1}}{\ell^{tk}B_k\left(1-\frac{1}{\ell^k}\right)} \times \\ \left(\sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ d>0}} d^{k-1}\delta\left(\frac{d}{\ell^t}\right)\ell\right)q^n - \sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ d>0}} d^{k-1}\delta\left(\frac{d}{\ell^{t-1}}\right)\right)q^n\right). \end{aligned} \tag{27}$$

Making the change of variables $d \to d\ell^t$ gives us

$$\begin{aligned} \sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ d>0}} d^{k-1}\delta\left(\frac{d}{\ell^t}\right)\ell\right)q^n &= \ell^{t(k-1)+1}\sum_{n\geq 1}\left(\sum_{\substack{d\mid \frac{n}{\ell^t}\\ d>0}} d^{k-1}\right)q^n \\ &= \ell^{t(k-1)+1}\sum_{n\geq 1}\sigma_{k-1}\left(\frac{n}{\ell^t}\right)q^n, \end{aligned} \tag{28}$$

and similarly, making the change of variables $d \to d\ell^{t-1}$ gives us

$$\sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ d>0}} d^{k-1}\delta\left(\frac{d}{\ell^{t-1}}\right)\right)q^n = \ell^{(t-1)(k-1)}\sum_{n\geq 1}\sigma_k\left(\frac{n}{\ell^{t-1}}\right)q^n, \tag{29}$$

where $\sigma_k\left(\frac{a}{b}\right) = 0$ if a and b are integers with $b \neq 0$ but $\frac{a}{b} \notin \mathbb{Z}$. Substituting (28) and (29) in (27) yields

$$E_{N,k,\chi}(z) = 1 - \frac{2k}{B_k(\ell^k-1)}\sum_{n\geq 1}\left(\ell^k\sigma_{k-1}\left(\frac{n}{\ell^t}\right) - \sigma_{k-1}\left(\frac{n}{\ell^{t-1}}\right)\right)q^n. \tag{30}$$

$E_{\ell^t,k,\chi}(z) \equiv 1 \pmod{\ell}$ since $\operatorname{ord}_\ell\left(\frac{2k}{B_k(\ell^k-1)}\right) \geq 1$ by Theorem 4.1. This proves Theorem 1.2.1.

Next, we assume $\ell = 2$. If $\operatorname{ord}_2(k) = j$, then

$$k = 2^j m = \phi(2^{j+1})m \tag{31}$$

for some positive odd integer m. Moreover, Theorem 4.1 implies that $\mathrm{ord}_2(B_k) = -1$ for every positive even integer $k \geq 4$. We examine $\mathrm{ord}_2(a(c_N))$ in two cases.

CASE $1'$. $2 \nmid N$.

$\mathrm{ord}_2(p^k - 1) \geq j + 1$ for every prime $p \mid N$ using (31), so

$$\mathrm{ord}_2(a(c_N)) \leq 1 + (j+1)(1 - \omega(N))$$

using (24). If $\omega(N) > 1$, it follows that $E_{N,k,\chi}(z) \not\equiv 1 \pmod 2$. Furthermore, if $N = p^b$ for some odd prime p and positive integer b, and if $\mathrm{ord}_2(p^k - 1) > j + 1$, then $E_{p^b,k,\chi}(z) \not\equiv 1 \pmod 2$.

CASE $2'$. $2 \mid N$.

Using (31) and (24) as in case $1'$, we obtain

$$\mathrm{ord}_2(a(c_N)) \leq 1 + (j+1)(2 - \omega(N)).$$

If $\omega(N) > 2$, then $E_{N,k,\chi}(z) \not\equiv 1 \pmod 2$. Also, if $N = 2^a p^b$ for some odd prime p and positive integers a and b, and if $\mathrm{ord}_2(p^k - 1) > j + 1$, then $E_{2^a p^b,k,\chi}(z) \not\equiv 1 \pmod 2$.

Hence, $E_{N,k,\chi}(z) \not\equiv 1 \pmod 2$ if N does not have the form $2^a p^b$, where p is an odd prime satisfying

$$\mathrm{ord}_2(p^k - 1) = j + 1 \tag{32}$$

and a and b are nonnegative integers. We therefore examine $E_{N,k,\chi}(z) \pmod 2$ when N does have this form.

In the case where $N = 2^a$ for some positive integer a and in the case where $N = p^b$ where p is an odd prime satisfying (32) and b is a positive integer, the formulas for $E_{N,k,\chi}(z)$ are given by (30) with $\ell = 2$ and $\ell = p$, respectively. The reasoning used there can also be used to show that $E_{N,k,\chi}(z) \equiv 1 \pmod 2$ in these cases.

In the case where $N = 2^a p^b$ for some odd prime p satisfying (32) and a and b positive integers, a formula for $E_{2^a p^b,k,\chi}(z)$ can be found by applying the same reasoning used to derive (30):

$$E_{2^a p^b,k,\chi}(z) = 1 - \frac{2k}{B_k(p^k - 1)(2^k - 1)} \sum_{n \geq 1} A(n) q^n,$$

where

$$A(n) = (2p)^k \sigma_{k-1}\left(\frac{n}{2^a p^b}\right) - p^k \sigma_{k-1}\left(\frac{n}{2^{a-1} p^b}\right) - 2^k \sigma_{k-1}\left(\frac{n}{2^a p^{b-1}}\right) + \sigma_{k-1}\left(\frac{n}{2^{a-1} p^{b-1}}\right).$$

Now observe that $E_{2^a p^b,k,\chi}(z) \equiv 1 \pmod 2$ since $\mathrm{ord}_2\left(\frac{2k}{B_k(p^k-1)(2^k-1)}\right) = 1$ using Theorem 4.1. This completes the proof of Theorem 1.2.

5. The Proof of Theorem 1.3

The proof of Theorem 1.3 follows by applying the Theorems of Carlitz (Theorems 5.1, 5.2) regarding the divisibility properties of the generalized Bernoulli numbers $B_{k,\chi}$ to the formula (4). We extend the definition of ord to rings of integers of number fields in the obvious way.

Theorem 5.1 (Carlitz) [**C, Thm.1**]. *Suppose that k is a positive integer, ℓ is a rational prime, and χ is a nontrivial primitive Dirichlet character with conductor N. Then*

$$\frac{B_{k,\chi}}{k} = \frac{\mathfrak{N}}{\mathfrak{D}},$$

where $\mathfrak{N}$ and $\mathfrak{D}$ are elements in O_{K_χ} with $\gcd(\mathfrak{N}, \mathfrak{D}) = 1$. *If N has at least two distinct rational prime divisors, then $\mathfrak{D} = 1$. If $N = \ell^t$, then $\mathfrak{D}$ is a product of prime divisors of ℓ.*

We denote the series in formula (4) by $E_{N,k,\chi}(z) = \sum_{n\geq 0} b(n)q^n$, and observe that

$$b(1) = \frac{-2k}{B_{k,\chi}}.$$

If N has at least two rational prime divisors, then there is an $\mathfrak{N} \in O_{K_\chi}$ for which $b(1) = \frac{-2}{\mathfrak{N}}$ by Theorem 5.1. It follows that if $\mathfrak{a}$ is an ideal in $O_{K_\chi,N}$ and $\mathfrak{a} \nmid (2)$, then $\mathrm{ord}_{\mathfrak{a}}(b(1)) \leq 0$, so that $E_{N,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$, proving Theorem 1.3.1. The proofs of Theorems 1.3.2-1.3.5 follow from Theorem 5.2.

Theorem 5.2 (Carlitz) [**C, Thm.4**]. *Suppose χ is a nontrivial primitive character with conductor N, and ℓ is an odd rational prime. Then the following are true.*

1. *If $N = \ell$, then $\frac{B_{k,\chi}}{k} \in \mathbb{Z}$ unless there is a primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$ for which*

$$\mathfrak{p} = \gcd(\ell, 1 - \chi(g)g^k) \neq (1), \tag{33}$$

where $\mathfrak{p}$ is an ideal in O_{K_χ}. In this case,

$$\ell B_{k,\chi} + 1 \equiv 0 \pmod{\mathfrak{p}^{1+\mathrm{ord}_\ell(k)}}.$$

2. *If $N = \ell^t$ for some integer $t \geq 2$, then $\frac{B_{k,\chi}}{k} \in \mathbb{Z}$ unless there is a primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$ for which*

$$\mathfrak{p} = \gcd(\ell, 1 - \chi(g)g^k) \neq (1),$$

where $\mathfrak{p}$ is an ideal in O_{K_χ}. In this case,

$$(1 - \chi(1+\ell))\frac{B_{k,\chi}}{k} \equiv 1 \pmod{\mathfrak{p}}.$$

3. *If $N = 4$, then*

$$\frac{B_{k,\chi}}{k} \equiv \begin{cases} 1 \pmod{\frac{1}{2}} & \text{if } k \text{ odd}, \\ 0 \pmod{1} & \text{if } k \text{ even}. \end{cases}$$

4. *If $N = 2^t$, for some integer $t \geq 3$, then $\frac{B_{k,\chi}}{k} \in \mathbb{Z}$.*

Note that if (33) holds, and if we let $B_{k,\chi} = \frac{U_{k,\chi}}{V_{k,\chi}}$, where $U_{k,\chi}$ and $V_{k,\chi} \in O_{K_\chi}$ and $\gcd(U_{k,\chi}, V_{k,\chi}) = 1$, then

$$\mathrm{ord}_{\mathfrak{p}}(B_{k,\chi}) = -\mathrm{ord}_{\mathfrak{p}}(\ell) \leq -1. \tag{34}$$

We now proceed with the proofs of Theorems 1.3.2-1.3.5. We assume in all cases that ℓ is an odd rational prime, $\mathfrak{a}$ is an ideal in $O_{K_\chi,N}$ with the property that $\mathfrak{a} \nmid (2)$, χ is a nontrivial primitive Dirichlet character, and $k \geq 3$ is an integer satisfying $\chi(-1) = (-1)^k$.

Proof (of Theorem 1.3.2). If $N = 4$ and k is even, then χ is trivial. If $N = 4$ and k is odd, then $b(1) = \frac{-4}{s}$ for some nonzero integer s by Theorem 5.2.3. If $\mathfrak{a} \nmid (4)$, then $\mathrm{ord}_{\mathfrak{a}}(b(1)) \leq 0$, so $E_{4,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$. If $N = 2^t$ for some integer $t \geq 3$, then $b(1) = \frac{-2}{j}$ for some nonzero integer j by Theorem 5.2.4. Hence, $\mathrm{ord}_{\mathfrak{a}}(b(1)) \leq 0$, so $E_{2^t,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$.

Proof (of Theorem 1.3.3). If $N = \ell$ and $\gcd(\ell, 1-\chi(g)g^k) = (1)$ for every primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$, then $b(1) = \frac{-2}{j}$ for some nonzero integer j by Theorem 5.2.1. Hence, $\mathrm{ord}_{\mathfrak{a}}(b(1)) \leq 0$, so $E_{\ell,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$. If there is a primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$ for which $\mathfrak{p} = \gcd(\ell, 1-\chi(g)g^k) \neq (1)$, then $\mathrm{ord}_{\mathfrak{p}}(B_{k,\chi}) = -\mathrm{ord}_{\mathfrak{p}}(\ell) \leq -1$ by (34). It follows that $\mathrm{ord}_{\mathfrak{p}}(b(n)) \geq 1$ for every $n \geq 1$ by formula (4), and thus, $E_{\ell,k,\chi}(z) \equiv 1 \pmod{\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\ell)}}$.

Proof (of Theorem 1.3.4). Suppose $N = \ell$ and $\chi = \left(\frac{\cdot}{\ell}\right)$, the Legendre symbol. If we choose an arbitrary primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$ and suppose that there is an $a \in \mathbb{Z}/\ell\mathbb{Z}$ for which $a^2 \equiv g \pmod{\ell}$, then $g^{\frac{\ell-1}{2}} \equiv a^{\ell-1} \equiv 1 \pmod{\ell}$ since $\gcd(a, \ell) = 1$. This contradicts the hypothesis that g is a primitive root of $\mathbb{Z}/\ell\mathbb{Z}$, so $\left(\frac{g}{\ell}\right) = -1$. Using this fact, observe that $\gcd(\ell, 1 - \left(\frac{g}{\ell}\right)g^k) = \gcd(\ell, 1 + g^k) \neq (1)$ if and only if $g^k \equiv -1 \pmod{\ell}$, i.e., if and only if $k \equiv \frac{\ell-1}{2} \pmod{\ell - 1}$. In this case $\gcd(\ell, 1 - \left(\frac{g}{\ell}\right)g^k) = (\ell)$, so $\mathrm{ord}_{\ell}\left(B_{k,\left(\frac{\cdot}{\ell}\right)}\right) \leq -1$ by (34). Using formula (4), it follows that $\mathrm{ord}_{\ell}(b(n)) \geq 1$ for all $n \geq 1$ if and only if $k \equiv \frac{\ell-1}{2} \pmod{\ell - 1}$.

Proof (of Theorem 1.3.5). If $N = \ell^t$ for some integer $t \geq 2$, and if $\gcd(\ell, 1-\chi(g)g^k) = (1)$ for every primitive root g of $\mathbb{Z}/\ell\mathbb{Z}$, then $b(1) = \frac{-2}{j}$ for some nonzero integer j by Theorem 5.2.2. Hence, $\mathrm{ord}_{\mathfrak{a}}(b(1)) \leq 0$, so $E_{\ell^t,k,\chi}(z) \not\equiv 1 \pmod{\mathfrak{a}}$.

Q.E.D.

References

[A] T. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.

[C] L. Carlitz, *Arithmetic Properties of Generalized Bernoulli Numbers*, J.Reine und Angew. Math. **202** (1959), 173-182.

[Ir-R] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1990 GTM 84.

[Iw] K. Iwasawa, *Lectures on p-Adic L-Functions*, Princeton Univ. Press, 1972.

[K] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, New York, 1984.

[Sc] B. Schoeneberg, *Elliptic Modular Functions*, Springer-Verlag New York Heidelberg Berlin, 1974.

[Sw-D] H.P.F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms, Modular functions of one variable*, Springer Lect. Notes in Math. **350** (1973), 1-55.

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