Math 544. Exam 2. 3/15/06. Name: ________________

- Read problems carefully. Show all work.
- No calculators or notes.
- The exam is approximately 15 percent of the total grade.
- There are 100 points total. Partial credit may be given.
1. (10 points)
   Give the following definitions. Be precise.
   (a) Define what it means for a set of vectors $S$ to span a subspace $W$.

   (b) Define what it means for a map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be a linear transformation.
2. (15 points)  
Let $A \in \text{Mat}_{3 \times 4}(\mathbb{R})$ be given as follows:

$$A = \begin{pmatrix} 
1 & 2 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 4 & 7 & 9 
\end{pmatrix}.$$  

(a) Find a basis for the null space of $A$.  
(b) What is the rank of $A$?
3. (10 points)

Suppose that $A \in \text{Mat}_{7 \times 4}(\mathbb{R})$.

(a) Explain why the rows of $A$ must be linearly dependent.

**Hint:** Compare the number of rows of $A$ to the largest possible value of the rank of $A$.

(b) Suppose that the nullity of $A$ is zero. Explain why the columns of $A$ must be linearly independent.
4. **(15 points)**

The set

\[ S = \left\{ s_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ s_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ s_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\} \]

is an **orthogonal** set (a fact you may assume).

Express the vector \( v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) as a linear combination of vectors in \( S \).
5. (10 points)

Suppose that $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a linear transformation with

$$
T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 11 \\ 9 \end{pmatrix}.
$$

Find the matrix $A \in \text{Mat}_{3 \times 3}(\mathbb{R})$ such that for all $x \in \mathbb{R}^3$, $T(x) = Ax$.

**Hint:** Write $T \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ in terms of $T(e_1), T(e_2), \text{ and } T(e_3)$ ($e_1, e_2, e_3$ are the standard basis vectors in $\mathbb{R}^3$). Now, solve for $T(e_3)$. 


6. (20 points)
Consider the matrix $A \in \text{Mat}_{2 \times 2}(\mathbb{R})$ given by

$$A = \begin{pmatrix} -8 & -5 \\ 6 & 3 \end{pmatrix}. $$

(a) Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $A$. Find $\lambda_1$ and $\lambda_2$.

(b) Find all the eigenvectors of $A$ associated to either $\lambda_1$ or $\lambda_2$ (your choice).
7. **(15 points)**

Let \( A \in \text{Mat}_{3 \times 3}(\mathbb{R}) \) be given by

\[
A = \begin{pmatrix}
2 & 3 & 4 \\
1 & 2 & 4 \\
1 & 2 & 1
\end{pmatrix}.
\]

- Compute \( \det A \) by expanding **down the second column**.
- Is \( A \) invertible? Why or why not?
- What is \( \det(A^3) \)?


8. **Bonus:** Attempt this problem if and only if you have completed 1-7. You will not be penalized if you do not attempt it.

Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$. Suppose that the entries in each row of $A$ add to zero. Show that $\det A = 0$.

**Hint:** Can you think of any vectors $x \in \mathbb{R}^n$ for which $Ax = \theta$?