1. Let \( \mathbb{Z} = \{ x \in \mathbb{Z} : x = 2k, k \in \mathbb{Z} \} \) for even integers.
\[ \mathbb{Z} + 1 = \{ x + 1 : x \in \mathbb{Z} \} \] for odd integers.

\((\mathbb{Z}, +)\) is a group. It satisfies:
- **Closure**: \( 2i + 2j = 2(i + j) \in \mathbb{Z} \) in \( \mathbb{Z} \)
- **Associativity**: Since \((\mathbb{Z}, +)\) is associative.
- **Identity**: \( 0 \in \mathbb{Z} \)
- **Inverses**: \( a + (-a) = 0 \in \mathbb{Z} \)

\((\mathbb{Z} + 1, +)\) is not a group. It satisfies:
- **Associativity**: Since \((\mathbb{Z}, +)\) is associative.
- **Inverses**: \( 2i + 1 + (-2i - 1) = 0 \in \mathbb{Z} + 1 \)

However, it does not satisfy:
- **Closure**: \( (2i + 1) + (2j + 1) = 2(i + j + 1) \notin \mathbb{Z} + 1 \) in \( \mathbb{Z} + 1 \)
- **Identity**: \( 0 \notin \mathbb{Z} + 1 \).
Claim \((\mathbb{Z}, \ast)\) is not a group.

Proof: It has no identity. To see this, suppose that \(ce \in \mathbb{Z}\) such that \(\forall a \in \mathbb{Z}\), we have \(a \ast c = \max(a, c) = a\).

Then \(c\) must be the smallest element of \(\mathbb{Z}\). But \(\mathbb{Z}\) has no smallest element.

Note, however, that \((\mathbb{Z}, \ast)\) is closed and associative:

\[
\forall a, b, c \in \mathbb{Z},
\begin{align*}
a \ast (b \ast c) &= \max (a, \max(b, c)) = \max \left( \max(a, b), c \right) = \max(a, \max(b, c)) \\
(a \ast b) \ast c &= \max (\max(a, b), c) = \max (a, \max(b, c))
\end{align*}
\]

2c. \(\forall a, b \in \mathbb{Z}, a \ast b = a - b\).

Claim: \((\mathbb{Z}, \ast)\) is not a group.

Proof: It is not associative: \(\forall a, b, c \in \mathbb{Z}\), we have

\[
\begin{align*}
(a \ast (b \ast c)) &= a - (b - c) = a - b + c \\
((a \ast b) \ast c) &= (a - b) - c = a - b - c
\end{align*}
\]

Moreover, it lacks an identity. Suppose that \(e \in \mathbb{Z}\) such that \(\forall a \in \mathbb{Z}, a \ast e \in \mathbb{Z}\). But \(e - a = a \leq e = 2a\).

However, \((\mathbb{Z}, \ast)\) is closed: \(\forall a, b \in \mathbb{Z}, a \ast b = a - b \in \mathbb{Z}\).
2d. \( \forall a, b \in \mathbb{Z}, \ a \ast b = 1ab1 \).

**Claim:** \((\mathbb{Z}, \ast)\) is not a group.

**Proof:** It lacks an identity element. To see this, suppose that \( e \in \mathbb{Z} \) such that \( \forall a \in \mathbb{Z}, \ a \ast e = 1ae1 = 1a1 \cdot 1e1 = a \).

Now if \( a < 0 \), then \( 1a1 = -a \). It follows that \( a \ast e = -a \cdot 1e1 = a \frac{-a}{e} \cdot 1e1 = -1 \). But \( 1e1 > 0 \).

Note, however, that it is closed: \( \forall a, b \in \mathbb{Z}, \ a \ast b = 1ab1 \in \mathbb{Z} \).

It is also associative: \((a \ast b) \ast c = 1ab1 \ast c = 1ab1 \cdot 1c1 = 1abc1\)

\[ a \ast (b \ast c) = a \ast 1b1 \cdot 1c1 = 1a1 \cdot 1b1 \cdot 1c1 = 1abc1. \]
3. Let \((G, \cdot)\) be a group. \(\forall a, b \in G\), define \(a \ast b = b \cdot a\).

(a) Show that \((G, \ast)\) is a group.

**Proof:**

- **Identity.** Let \(e\) be the identity for \((G, \cdot)\). We claim that \(e\) is also the identity for \((G, \ast)\). Let \(a \in G\). Then

  \[a \ast e = e \cdot a = a,\] since \(e\) is the identity for \((G, \cdot)\); hence it is also the identity for \((G, \ast)\).

- **Associativity.** Let \(a, b, c \in G\). Then

  \[(a \ast b) \ast c = c \cdot (a \ast b) = c \cdot (b \cdot a) = (c \cdot b) \cdot a = a \ast (c \cdot b) = a \ast (b \ast c),\]

  \(\uparrow\) is associativity.

- **Closure.** Let \(a, b \in G\). Then \(a \ast b = b \cdot a \in G\) since \((G, \cdot)\) is closed.

- **Inverses.** Let \(a \in G\), and let \(b \in G\) be the inverse for \(a\) under \(\cdot\) then

\[a \ast b = b \cdot a = e = a \cdot b = b \ast a,\]

\(\uparrow\) inverse under \(\ast\).

(b) \((G, \ast) = (G, \cdot) \iff (G, \cdot)\) is commutative. In particular,

\((G, \cdot)\) is commutative \(\iff \forall a, b \in G\), \(a \cdot b = b \cdot a = a \ast b\)

\(\iff \ast\) and \(\cdot\) coincide \(\iff (G, \ast) = (G, \cdot)\).

Example: \((\mathbb{Z}_n, \ast) = (\mathbb{Z}_n, \cdot); (S_3, \ast) \neq (S_3, \cdot).\)
6. \[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\
2 & 2 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
3 & 3 & 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 \\
4 & 4 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 \\
5 & 5 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 6 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 7 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
(\mathbb{Z}_8, +)
\]

7. \[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 4 & 6 & 1 & 3 & 5 \\
3 & 3 & 6 & 2 & 5 & 1 & 4 \\
4 & 4 & 1 & 5 & 2 & 6 & 3 \\
5 & 5 & 3 & 1 & 6 & 4 & 2 \\
6 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
(\mathbb{Z}_7^*, \times)
\]
9. Let \( G = \{ x \in \mathbb{R} : x > 0 \text{ and } x \neq 1 \} \). For \( a, b \in G \), define 
\[ a \ast b = a^{\ln b} \]. Prove that \( (G, \ast) \) is an abelian group.

**Proof:**

- **Closure.** Let \( a, b \in G \). \( a > 0 \Rightarrow a^{\ln b} > 0 \)
  \[ b > 0 \Rightarrow a^{\ln b} \neq 1 \]
  \[ a, b \in G \]

- **Associativity.** Let \( a, b, c \in G \). Then
  \[ (a \ast b) \ast c = (a^{\ln b})^{\ln c} = a^{\ln b \cdot \ln c} = a^{\ln (b^{\ln c})} = a^{\ln (c^{-1})} = a \ast (b \ast c) \]

- **Identity.** The number 1 (with \( \ln 1 = 0 \)) is the identity for \( \ast \).
  Let \( a \in G \). Then \( a \ast 1 = a^{\ln 1} = a^1 = a \) and
  \[ 1 \ast a = e^{\ln a} = a \]

- **Inverse.** Let \( a \in G \). Then
  \[ e^{\ln a} \ast a = (e^{\ln a})^{\ln a} = e \]
  \[ a \ast e^\frac{1}{\ln a} = a^{\ln (e^{\frac{1}{\ln a}})} = a^{\frac{1}{\ln a}} = a \]
  \[ e \ast a = e^{\ln a} = a \]
  Hence, \( a^{-1} = e^{\frac{1}{\ln a}} \).

- **Commutativity.** Let \( a, b \in G \). Then
  \[ a \ast b = a^{\ln b} = e^{\ln (a^{\ln b})} = e^{(\ln b)(\ln a)} = e^{(\ln b)(\ln a)} \]
  \[ b \ast a = b^{\ln a} = e^{\ln (b^{\ln a})} = e^{(\ln a)(\ln b)} \]
  \[ a \ast b = b \ast a \]
10. Show that the set \( A = \{ f_{m, b} : \mathbb{R} \to \mathbb{R} : m \neq 0 ; f_{m, b}(x) = mx + b \} \)

is a group under composition.

**Proof**

- **Closure.** Let \( f_{m_1, b_1}(x), f_{m_2, b_2}(x) \in A. \) Then
  \[
  (f_{m_1, b_1} \circ f_{m_2, b_2})(x) = f_{m_1, b_1}(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_2m_1x + m_1b_2 + b_1 \in A.
  \]

- **Associativity.** This holds since function composition is associative.

- **Identity.** \( f_{1, 0}(x) \) is the identity. Let \( f_{m, b}(x) \in A. \) Then
  \[
  (f_{1, 0} \circ f_{m, b})(x) = f_{m, b}(x) = (f_{m, b} \circ f_{1, 0})(x).
  \]

- **Inverses.** Let \( f_{m, b}(x) \in A. \) Then \( f_{m^{-1}, -b/m}(x) \) is its inverse.
  \[
  (f_{m^{-1}, -b/m} \circ f_{m, b})(x) = \frac{x}{m}(mx + b) = \frac{1}{m} = x = f_{1, 0}(x)
  \]
  \[
  (f_{m, b} \circ f_{m^{-1}, -b/m})(x) = m\left(\frac{x}{m} - \frac{b}{m}\right) + b = x = f_{1, 0}(x).
  \]
11. Show that \( S = \{ (\begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix}) \in M_2(\mathbb{R}) : m \neq 0 \} \) is a group under multiplication.

Proof:

- Closure: Let \( \begin{pmatrix} m_1 & b_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m_2 & b_2 \\ 0 & 1 \end{pmatrix} \in S. \) Then

\[
\begin{pmatrix} m_1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m_2 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_1m_2 & m_1b_2+b_1 \\ 0 & 1 \end{pmatrix} \in S.
\]

- Associativity: This holds since multiplication in \( M_2(\mathbb{R}) \) is associative.

- Identity: \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S \) is the identity.

- Inverse: Let \( \begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix} \in S. \) Then \( \begin{pmatrix} 1/m & -b/m \\ 0 & 1 \end{pmatrix} \in S \) is its inverse.

\[
\begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/m & -b/m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1/m & -b/m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
13. Let $S = \mathbb{R} \setminus \{-1\}$. For $a, b \in S$, define $a \star b = a + b + ab$.

Show that $(S, \star)$ is a group.

**Proof:**

**Closure.** Let $a, b \in S$. Note that $a + b + ab \in \mathbb{R}$. Also, we have

\[
a + b + ab = -1 \implies a(1+b) = -(1+b) \implies a = -1
\]

Hence, $a \star b \in S$.

**Associativity.** Let $a, b, c \in S$. Then

\[
(a \star b) \star c = (a + b + ab) \star c = a + b + ab + c + (a(1+b)c) = a + b + c + ab + ac + (a+b)c.
\]

\[
= a + b + c + bc + a(b + c + bc)
\]

**Identity.** $0 \in S$ is the identity. Let $a \in S$. Then

\[
a \star 0 = a + 0 + a \cdot 0 = a
\]

\[
0 \star a = 0 + a + 0 \cdot a = a.
\]

**Inverse.** Let $a \in S$. Then $-\frac{a}{1+a} \in S$ is its inverse.

\[
a \star \left(-\frac{a}{1+a}\right) = a + \left(-\frac{a}{1+a}\right) + a\left(-\frac{a}{1+a}\right) = a - \left(\frac{a^2 + a}{a+1}\right) = a - a \left(\frac{a+1}{a+1}\right) = 0
\]

\[
\left(-\frac{a}{1+a}\right) \star a = -\frac{a}{1+a} + a + \left(-\frac{a}{1+a}\right) = a + \left(-\frac{a}{1+a}\right) + a\left(-\frac{a}{1+a}\right) = 0.
\]
23. Let $G$ be a group. Prove: $x^2 = e \forall x \in G \Rightarrow G$ is abelian.

**Proof:** Suppose that $\forall x \in G$, we have $x^2 = e$. We will show that $\forall y, z \in G$, we have $yz = zy$.

First, observe that $y, z \in G \Rightarrow yz \in G \Rightarrow (yz)^2 = yzyz = e$

by the hypothesis

$\Rightarrow yz = z^{-1}y^{-1}$.

Next, note also that the hypothesis $\Rightarrow y^2 = e$, $\Rightarrow y = y^{-1}$,

$z^2 = e \Rightarrow z = z^{-1}$.

Substitute $y = y^{-1}, z = z^{-1}$ in $yz = z^{-1}y^{-1}$ to obtain

$yz = z^{-1}y^{-1} = zy$. Hence, $G$ is abelian.

24. Let $G$ be a finite group with an even $\# \text{ of elements.}$

Show that $\exists a \neq e$ in $G$ with $a^2 = e$.

**Proof:** Observe that $a \neq e$ in $G$ has $a^2 = e \Leftrightarrow a = a^{-1}$.

Hence, it suffices to show that $G$ has an element $a \neq e$ which is its own inverse. Suppose that this is not the case. Then $\forall a \neq e$ in $G$, we have $a \neq a^{-1}$. Hence, we can list all elements of $G$ (excluding $e$) in pairs: \{a_1, a_1^{-1}\}, \{a_2, a_2^{-1}\}, \ldots, \{a_n, a_n^{-1}\}. I.e., the set $G \setminus \{e\}$ has $2n$ elements. But then $G$ has $2n + 1$ elements, which is a contradiction since $G$ has an even number of elements.