Math 546, Final Exam Information.
Tuesday, December 8, 9 - 12 pm, LC 303B.

The Final Exam will be based on:

- Sections 2.1 - 2.3, 3.1 - 3.8, 7.2.
- The corresponding assigned homework problems (see http://www.math.sc.edu/~boylan/SCCourses/546Fa09/546.html). At minimum, you need to understand how to do the homework problems.

Useful materials:

- Exams 1, 2, 3 and their solutions.
- Homeworks 1 – 11 and their solutions.

New Topic List (not necessarily comprehensive): (Consult review handouts for Exams I, II, III for a list of old topics.)

You will need to know how to define vocabulary words/phrases defined in class.

§7.2 Conjugacy.

Definition (center). Let $G$ be a group. The center of $G$ is

$$Z(G) = \{x \in G : \forall y \in G, xy = yx\}.$$ 

Properties:

1. $Z(G)$ is a normal subgroup of $G$.
2. $G$ is abelian $\iff Z(G) = G$.
3. Examples: Let $n \geq 1$.
   - If $n \geq 3$, then we have $Z(S_n) = \langle e \rangle$.
   - We have
     $$Z(D_n) = \begin{cases} 
     \langle e \rangle & \text{if } n \text{ is odd,} \\
     \langle a^{n/2} \rangle = \{e, a^{n/2} \} & \text{if } n \text{ is odd.}
     \end{cases}$$
Definition (conjugacy). Let $x, y \in G$. Then $y$ is conjugate to $x$ $\iff$ there exists $a \in G$ with $y = axa^{-1}$.

Properties:

1. Conjugacy is an equivalence relation on $G$.
2. The equivalence classes are called conjugacy classes.
3. Let $x \in G$. The conjugacy class of $x$ is
   \[ \mathcal{C}(x) = \{axa^{-1} : a \in G \} \]
4. Let $x \in G$. Then $\mathcal{C}(x) = \{x\}$ (\(\mathcal{C}(x)\) is trivial) $\iff$ $x \in Z(G)$.

Definition (centralizer). Let $x \in G$. Then the centralizer of $x$ in $G$ is
\[ C(x) = \{a \in G : axa^{-1} = x\} \]

Properties:

1. Let $x \in G$. Then $C(x)$ is a subgroup of $G$.
2. $C(x)$ has subgroups $\langle x \rangle$ and $Z(G)$.
3. Let $x \in G$. Then $C(x) = G \iff x \in Z(G)$.

Key Fact: Let $x \in G$. Then we have
\[ |\mathcal{C}(x)| = \frac{|G|}{C(x)} \]

Theorem (Class equation): If $G$ is a finite group, then we have
\[ |G| = |Z(G)| + \sum_x |\mathcal{C}(x)| = |Z(G)| + \sum_x \frac{|G|}{C(x)} \]
The sum is taken over $x \in G$ for which $\mathcal{C}(x)$ are non-trivial and disjoint.

Examples:

1. If $G$ is abelian, then $\forall x \in G$, we have $C(x) = G$ and $\mathcal{C}(x) = \{x\}$.
2. Let $n \geq 1$. The centralizers and conjugacy classes in $G = D_n = \{e, a, \ldots a^{n-1}, b, ba, \ldots ba^{n-1}\}$ are as follows.
• If $n$ is odd, then $\mathclap{\text{C}}\ell(e) = \{e\}$; for all $1 \leq i \leq n - 1$, $\mathclap{\text{C}}\ell(a^i) = \{a^i, a^{n-i}\}$; for all $0 \leq j \leq n - 1$, $\mathclap{\text{C}}\ell(ba^j) = \{b, ba, \ldots, ba^{n-1}\}$. The class equation is

$$|D_n| = 2n = 1 + 2 \left( \frac{n-1}{2} \right) + n.$$  

The centralizers are: $C(e) = G$; for all $1 \leq i \leq n - 1$, $C(a^i) = \langle a \rangle$; for all $0 \leq j \leq n - 1$, $C(ba^j) = \langle ba^j \rangle$.

• If $n$ is even, then $\mathclap{\text{C}}\ell(e) = \{e\}$; $\mathclap{\text{C}}\ell(a^{n/2}) = \{a^{n/2}\}$; for all $1 \leq i \neq n/2 \leq n - 1$, $\mathclap{\text{C}}\ell(a^i) = \{a^i, a^{n-i}\}$; for all $1 \leq j$ even $\leq n - 1$, $\mathclap{\text{C}}\ell(ba^j) = \{b, ba^2, \ldots, ba^{n-2}\}$; for all $1 \leq k$ odd $\leq n - 1$, $\mathclap{\text{C}}\ell(ba^k) = \{ba, ba^3, \ldots, ba^{n-1}\}$. The class equation is

$$|D_n| = 2n = 2 + 2 \left( \frac{n-1}{2} - 1 \right) + \left( \frac{n}{2} \right) \cdot 2.$$  

The centralizers are $C(e) = C(a^{n/2}) = G$; for all $1 \leq i \neq n/2 \leq n - 1$, $C(a^i) = \langle a \rangle$; for all $1 \leq j \leq n - 1$, $C(ba^j) = \langle e, a^{n/2}, ba^j, ba^{j+n/2} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

3. Let $n \geq 1$, and let $\sigma \in S_n$ have factorization into disjoint cycles given by $\sigma = \tau_1 \cdots \tau_j$. Let $\ell_i$ be the length of the cycle $\tau_i$. Then the cycle type of $\sigma$ is $(\ell_1, \ldots, \ell_j)$.

**Key Fact:** $\sigma_1$ and $\sigma_2$ are conjugate in $S_n \iff$ they have the same cycle type. The number of conjugacy classes in $S_n$ is $p(n)$, the number of partitions of $n$.

**Consequences:**

1. Let $p$ be prime, and let $n \geq 1$. If $|G| = p^n$, then we have $Z(G) \neq \{e\}$. In particular, $|Z(G)| \geq p$; the center of $G$ is non-trivial.

2. Let $p$ be prime. If $|G| = p^2$, then $G$ is abelian.

3. **Cauchy’s Theorem:** Let $p$ be prime and suppose that $p \mid |G|$. Then there exists $x \in G$ with $o(x) = p$. Moreover, the subgroup $H = \langle x \rangle$ has order $p$. 

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