Exam 3 will be based on:

- Sections 4.1 - 4.5, 4.7, and 5.2.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/~boylan/SCCourses/math5442/544.html).
  At minimum, you need to understand how to do the homework problems.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/ phrases defined in class.

§4.1: The eigenvalue for $2 \times 2$ matrices: Definition and computation of eigenvalues and eigenvectors for $2 \times 2$ matrices.

§4.2: Determinants and the eigenvalue problem: Definition and computation of determinants of matrices $A \in \text{Mat}_{n \times n}$. Computation of determinants by expansion across rows or down columns using minors and cofactors. What is the minor and cofactor associated to a matrix entry $(a_{i,j})$ of $A$? Properties of determinants, for example:

- $\det(AB) = \det(A)\det(B)$.
- $A \in \text{Mat}_{n \times n}$ is singular $\iff$ $\det(A) = 0$.
- If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

What is the determinant of a triangular matrix?

§4.3: Elementary operations and determinants: Important property: $\det(A) = \det(A^T)$. Effects of elementary row and column operations on the computation of a determinant:

- Interchanging two rows or two columns changes the sign of the determinant.
- If the row operation
  \[ R_i \mapsto \frac{1}{k} R_i, \quad k \neq 0 \]
  transforms matrix $A$ into matrix $B$, then $\det(A) = k\det(B)$. (In effect, you are “factoring” $k$ out of the $i$th row of $A$.) Similarly, if the column operation
  \[ C_i \mapsto \frac{1}{k} C_i, \quad k \neq 0 \]
  transforms matrix $A$ into matrix $B$, then $\det(A) = k\det(B)$. 

A row operation of the form

\[ R_i \leftrightarrow R_i + kR_j, \; k \neq 0, \; i \neq j \]

does nothing to the determinant. Similarly, a column operation of the form

\[ C_i \leftrightarrow C_i + kC_j, \; k \neq 0, \; i \neq j \]

does nothing to the determinant.

\section{4.4: Eigenvalues and the characteristic polynomial}

The definition and computation of the \textbf{eigenvalues} of a matrix \( A \in \text{Mat}_{n \times n} \), i.e., computation of the \textbf{characteristic polynomial} \( p(t) = \det(A - tI_n) \); the \textbf{algebraic multiplicity} of an eigenvalue \( \lambda \) is the number of times the factor \((t - \lambda)\) occurs in the characteristic polynomial \( p(t) \).

- If \( \lambda \) is an eigenvalue of \( A \) and \( k \geq 0 \) is an integer, then \( \lambda^k \) is an eigenvalue of \( A^k \).
- If \( A \) is invertible and \( \lambda \) is an eigenvalue of \( A \), then \( \frac{1}{\lambda} \) is an eigenvalue of \( A^{-1} \).
- If \( \lambda \) is an eigenvalue of \( A \), then it is also an eigenvalue of \( A^T \).
- A matrix \( A \) has 0 as one of its eigenvalues if and only if it is singular.

What are the eigenvalues of a triangular matrix?

\section{4.5: Eigenspaces and eigenvectors}

The definition and computation of the eigenvectors of a matrix \( A \in \text{Mat}_{n \times n} \). If \( \lambda \) is an eigenvalue of \( A \in \text{Mat}_{n \times n} \), then the \textbf{eigenspace} associated to \( \lambda \) is \( E_\lambda = \text{Null}(A - \lambda I) \) and the \textbf{geometric multiplicity} of \( \lambda \) is the dimension of \( E_\lambda \) (i.e., the nullity of \( A - \lambda I \)). The relationship between algebraic and geometric multiplicities is

\[ 1 \leq \text{geometric mult.}(\lambda) \leq \text{algebraic mult.}(\lambda). \]

Definition of a \textbf{defective} matrix: a matrix \( A \) is defective if \( A \) has at least one eigenvalue whose geometric mult. is strictly less than its algebraic mult. i.e., there is an eigenvalue \( \lambda \) with

\[ \text{geom. mult.}(\lambda) < \text{alg. mult.}(\lambda). \]

Important fact: Eigenvectors associated to distinct eigenvalues are linearly independent. In particular if \( A \in \text{Mat}_{n \times n} \) has \( n \) distinct eigenvalues, then \( A \) has \( n \) linearly independent eigenvectors; i.e., \( \mathbb{R}^n \) has a basis consisting of eigenvectors for \( A \).

\section{4.7: Similarity transformations and diagonalization}

Matrices \( A \) and \( B \in \text{Mat}_{n \times n} \) are similar if there is an invertible matrix \( S \) for which

\[ B = S^{-1}AS. \]

A matrix \( A \) is diagonalizable if it is similar to a diagonal matrix \( B \). If \( A \) and \( B \) are similar, they have the same:

- characteristic polynomial
- eigenvalues (but the corresponding eigenvectors are typically different! If \( B = S^{-1}AS \) (so \( A \) and \( B \) are similar) and if \( x \) is an eigenvector of \( B \) associated to \( \lambda \) (so \( Bx = \lambda x \)), then \( Sx \) is an eigenvector of \( A \) associated to \( \lambda \) (so \( A(Sx) = \lambda(Sx)) \)).
Criterion for diagonalizability: the diagonalizability of \( A \) is equivalent to

- \( A \) has \( n \) linearly independent eigenvectors (the maximum possible).
- \( A \) is not defective (i.e., the geometric and algebraic multiplicities agree for all eigenvalues of \( A \)).

If \( A \) is diagonalizable, then there is an invertible matrix \( S \) and a diagonal matrix \( B \) for which

\[
B = S^{-1}AS.
\]

How do you find the matrices \( S \) and \( B \)?

- Compute the eigenvalues of \( A \) and their algebraic multiplicities. Suppose that the distinct eigenvalues of \( A \) are \( \lambda_1, \ldots, \lambda_k \).
- Compute bases for the eigenspaces \( E_{\lambda_1}, \ldots, E_{\lambda_k} \). The dimension of \( E_{\lambda_i} \) is the geometric multiplicity of \( \lambda_i \). If for all \( i \),

\[
\text{alg. mult.}(\lambda_i) = \text{geom. mult.}(\lambda_i),
\]

then \( A \) is diagonalizable.
- Form a set \( W = \{ \vec{w}_1, \ldots, \vec{w}_n \} \) consisting of all the basis vectors for the eigenspaces of \( A \). Then the invertible matrix \( S \) which diagonalizes \( A \) is

\[
S = (\vec{w}_1 \mid \vec{w}_2 \mid \cdots \mid \vec{w}_n).
\]

So we have

\[
B = S^{-1}AS,
\]

where \( B \) is a diagonal matrix with diagonal entry \((B)_{ii} = \lambda \) and \( \lambda \) is the eigenvalue of \( A \) associated to the eigenvector \( \vec{w}_i \): \( A\vec{w}_i = \lambda \vec{w}_i \).

If \( A \) is diagonalizable, and \( k \geq 0 \) is an integer, how can you compute \( A^k \)? Here’s how: \( A \) diagonalizable implies that for some invertible matrix \( S \), \( B = S^{-1}AS \) is diagonal. We then have \( B^k = (S^{-1}AS)^k = S^{-1}A^kS \). Moving the \( S \)'s to the left side, we obtain \( SB^kS^{-1} = A^k \). So if you know \( S \) and \( S^{-1} \) (it is easy to compute \( B^k \) if \( B \) is diagonal), you can compute \( A^k \).

5.2: **Vector spaces:** The definition of vector space (a set \( V \) and a scalar field \( F \) together with an addition operation on \( V \) and a scalar multiplication operation); in particular, the ten vector space axioms: 2 closure axioms, 4 axioms for vector addition, 4 axioms for scalar multiplication. Examples of vector spaces: \( \text{Mat}_{m \times n}(\mathbb{R}) \), \( P_n \). Check whether a set \( V \) together with an addition and scalar multiplication is or is not a vector space.