Math 544, Exam 2 Information

Exam 2 will be based on:

- Sections 3.2 - 3.7.
- The corresponding assigned homework problems (see http://www.math.sc.edu/~boylan/SCCourses/math5442/544.html). At minimum, you need to understand how to do the homework problems.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§3.2: Vector space axioms:

- closure properties (2)
- addition properties (4)
- multiplication properties (4)

Subspaces: How do you determine whether a subset $W$ of a vector space $V$ is a subspace?

1. To show that $W$ is a subspace, you need to verify the two subspace axioms.
2. To show that $W$ is not a subspace, it suffices to provide an example in which one of the axioms is violated.

§3.3: Examples of subspaces: What is the Span of a set of vectors?

Let $A \in \text{Mat}_{m \times n}$. Important subspaces associated to $A$ are:

- The null space of $A$ (a subspace of $\mathbb{R}^n$).
- The range of $A$ (a subspace of $\mathbb{R}^m$).
- The column space of $A$ (a subspace of $\mathbb{R}^m$ which coincides with the range of $A$).
- The row space of $A$ (a subspace of $\mathbb{R}^n$).

Problem: Given a matrix $A$, provide an algebraic description of the vectors in each of the above subspaces.

Fact: If $A$ and $B$ are row equivalent $m \times n$ matrices, then they have the same row space.

§3.4: Bases for subspaces: What is a basis for a subspace $W$ of $\mathbb{R}^n$?

Fact: If $B$ is a basis for $W$, then every vector $w \in W$ has a unique representation as a linear combination of vectors from $B$.

Given a matrix $A \in \text{Mat}_{m \times n}$, compute bases for the following subspaces:
• **Row**\((A)\): To compute a basis for Row\((A)\), put \(A\) in echelon form, \(B\). The nonzero rows of \(B\) form a basis for Row\((A)\).

• **Col**\((A)\): There are two methods.

  1. If you want a basis consisting of columns of \(A\), you put \(A\) in reduced echelon form, \(B\). The columns of \(B\) with the leading 1’s correspond to the columns of \(A\) which comprise the basis.

  2. If you do not require your basis to be a subset of the column vectors of \(A\), you can compute a basis for Col\((A)\) by computing a basis for Row\((A^\top) = \text{Col}(A)\). To do this, put \(A^\top\) in echelon form. The non-zero rows of the echelon form give the desired basis vectors.

• **Range**\((A)\): Since the range and column space of \(A\) agree, you compute Range\((A)\) just as you would Col\((A)\).

• **Null**\((A)\): Null\((A)\) is the set of solutions to the homogenous system \(A\vec{x} = \vec{0}\). Therefore, begin by solving \(A\vec{x} = \vec{0}\). Convert your solution to vector form. i.e., write your solution as Null\((A) = \text{Span}(B)\). Verify that the vectors in \(B\) are linearly independent.

§3.5: **Dimension**: Let \(W\) be a subspace of \(\mathbb{R}^n\).

**Facts:**

• \(W\) has many bases, but the number of vectors in every basis is the same. This number is called the **dimension** of \(W\).

• If \(A \in \text{Mat}_{m \times n}\), then \(\text{rank}(A) + \text{nullity}(A) = n\).

  (what is rank\((A)\)? nullity\((A)\)?)

• \(\text{rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A))\).

• \(A \in \text{Mat}_{n \times n}\) is non-singular if and only if \(\text{rank}(A) = n\) and \(\text{nullity}(A) = 0\).

**Know Theorem 3.9, page 207.**

§3.6: **Orthogonal bases for subspaces**: Define the terms **orthogonal** and **orthonormal**. How do you convert an orthogonal set to an orthonormal one? If \(B\) is a basis for a subspace \(W\) of \(\mathbb{R}^n\) consisting of orthogonal vectors, how do you express a vector \(\vec{w} \in W\) as a linear combination of basis vectors? Must a basis of orthogonal vectors for \(W\) always exist?

§3.7: **Linear transformations from** \(\mathbb{R}^n\) **to** \(\mathbb{R}^m\): Define what it means for a function \(T : \mathbb{R}^n \mapsto \mathbb{R}^m\) to be a linear transformation. Given a function \(T : \mathbb{R}^n \mapsto \mathbb{R}^m\), how does one check whether \(T\) is a linear transformation?

**Fact**: If \(T : \mathbb{R}^n \mapsto \mathbb{R}^m\) is a linear transformation, then for every \(\vec{x} \in \mathbb{R}^n\), \(T(\vec{x}) = A\vec{x}\), where

\[
A = [T(e_1) \mid \cdots \mid T(e_n)]
\]

(i.e., it suffices to know what \(T\) does to the standard basis for \(\mathbb{R}^n\)).

What are the **Null space** and **Range** of \(T\)?

2