Math 544, Exam 3 Information

Exam 3 will be based on:

- Sections 4.1 - 4.5, 4.7, and 5.2, 5.3.
- The corresponding assigned homework problems (see http://www.math.sc.edu/~boyland/SCCourses/math5443/544.html).
  At minimum, you need to understand how to do the homework problems.
- Quizzes: 7 - 10.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§4.1: The eigenvalue problem for $2 \times 2$ matrices: Definition and computation of eigenvalues and eigenvectors for $2 \times 2$ matrices.

§4.2: Determinants and the eigenvalue problem: Definition and computation of determinants of matrices $A \in \text{Mat}_{n \times n}(\mathbb{R})$. Computation of determinants by expansion across rows or down columns using minors and cofactors. What is the minor and cofactor associated to a matrix entry $(a_{ij})$ of $A$? Properties of determinants, for example:

- $\det(AB) = \det(A)\det(B)$.
- $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is singular $\iff \det(A) = 0$.
- If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

What is the determinant of a triangular matrix?

§4.3: Elementary operations and determinants: Important properties: $\det(A) = \det(A^T)$, $\det(cA) = c^n\det(A)$ for $c \neq 0$ in $\mathbb{R}$. Effects of elementary row and column operations on the computation of a determinant:

- Interchanging two rows or two columns changes the sign of the determinant.
- If the row operation
  \[ R_i \rightarrow \frac{1}{k}R_i, \quad k \neq 0 \]
  transforms matrix $A$ into matrix $B$, then $\det(A) = k\det(B)$. (In effect, you are “factoring” $k$ out of the $i$th row of $A$.) Similarly, if the column operation
  \[ C_i \rightarrow \frac{1}{k}C_i, \quad k \neq 0 \]
  transforms matrix $A$ into matrix $B$, then $\det(A) = k\det(B)$. 
• A row operation of the form
  \[ R_i \mapsto R_i + kR_j, \quad k \neq 0, \quad i \neq j \]
does nothing to the determinant. Similarly, a column operation of the form
  \[ C_i \mapsto C_i + kC_j, \quad k \neq 0, \quad i \neq j \]
does nothing to the determinant.

\section{4.4: Eigenvalues and the characteristic polynomial:} The definition and computation of the eigenvalues of a matrix \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \). i.e., computation of the characteristic polynomial \( p(t) = \det(A - tI_n) \); the algebraic multiplicity of an eigenvalue \( \lambda \) is the number of times the factor \((t - \lambda)\) occurs in the characteristic polynomial \( p(t) \).

  • If \( \lambda \) is an eigenvalue of \( A \) and \( k \geq 0 \) is an integer, then \( \lambda^k \) is an eigenvalue of \( A^k \).
  • If \( \lambda \) is an eigenvalue of \( A \) and \( \alpha \in \mathbb{R} \), then \( \lambda + \alpha \) is an eigenvalue of \( A + \alpha I_n \).
  • If \( A \) is invertible and \( \lambda \) is an eigenvalue of \( A \), then \( \frac{1}{\lambda} \) is an eigenvalue of \( A^{-1} \).
  • If \( \lambda \) is an eigenvalue of \( A \), then it is also an eigenvalue of \( A^T \).
  • A matrix \( A \) has 0 as one of its eigenvalues if and only if it is singular.

What are the eigenvalues of a triangular matrix?

\section{4.5: Eigenspaces and eigenvectors:} The definition and computation of the eigenvectors of a matrix \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \). If \( \lambda \) is an eigenvalue of \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \), then the eigenspace associated to \( \lambda \) is \( E_\lambda = \text{Null}(A - \lambda I) \) and the geometric multiplicity of \( \lambda \) is the dimension of \( E_\lambda \) (i.e., the nullity of \( A - \lambda I \)). The relationship between algebraic and geometric multiplicities is

\[ 1 \leq \text{geometric mult.}(\lambda) \leq \text{algebraic mult.}(\lambda). \]

Definition of a defective matrix: a matrix \( A \) is defective if \( A \) has at least one eigenvalue whose geometric mult. is strictly less than its algebraic mult. i.e., there is an eigenvalue \( \lambda \) with

\[ \text{geom. mult.}(\lambda) < \text{alg. mult.}(\lambda). \]

**Important fact:** Eigenvectors associated to distinct eigenvalues are linearly independent. As a consequence, if \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \) is not defective, then \( A \) has \( n \) linearly independent eigenvectors, and these eigenvectors form a basis for \( \mathbb{R}^n \). In particular,, if \( A \) has \( n \) distinct eigenvalues, then \( A \) is not defective.

\section{4.7: Similarity transformations and diagonalization:} Matrices \( A \) and \( B \in \text{Mat}_{n \times n}(\mathbb{R}) \) are similar if there is an invertible matrix \( S \) for which

\[ B = S^{-1}AS. \]
A matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$. If $A$ and $B$ are similar, they have the same:

- characteristic polynomial: $p_A(t) = p_B(t)$. **However, the converse is not true:** If $p_A(t) = p_B(t)$, then it is not always true that $A$ and $B$ are similar.
- eigenvalues and algebraic multiplicities (but the corresponding eigenvectors are typically different! If $B = S^{-1}AS$ (so $A$ and $B$ are similar) and if $\vec{x}$ is an eigenvector of $B$ associated to $\lambda$ (so $B\vec{x} = \lambda \vec{x}$), then $S\vec{x}$ is an eigenvector of $A$ associated to $\lambda$ (so $A(S\vec{x}) = \lambda(S\vec{x})$)).

**Criterion for diagonalizability:** The diagonalizability of $A$ is equivalent to

- $A$ has $n$ linearly independent eigenvectors (the maximum possible).
- $A$ is not defective (i.e., the geometric and algebraic multiplicities agree for all eigenvalues of $A$). So a matrix $A$ is either defective or diagonalizable.

If $A$ is diagonalizable, then there is an invertible matrix $S$ and a diagonal matrix $D$ for which

$$D = S^{-1}AS.$$  

How do you find the matrices $S$ and $D$?

- Compute the eigenvalues of $A$ and their algebraic multiplicities. Suppose that the distinct eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_k$.
- Compute bases $B_1, \ldots, B_k$ for the eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$. The dimension of $E_{\lambda_i}$ is the geometric multiplicity of $\lambda_i$. If for all $i$,

$$\text{alg. mult.}(\lambda_i) = \text{geom. mult.}(\lambda_i),$$

then $A$ is diagonalizable.
- If $A$ is diagonalizable, form the set $B = \{\vec{w}_1, \ldots, \vec{w}_n\}$ consisting of all the basis vectors for the eigenspaces of $A$. Then the invertible matrix $S$ which diagonalizes $A$ is

$$S = (\vec{w}_1 \mid \vec{w}_2 \mid \cdots \mid \vec{w}_n).$$

So we have

$$D = S^{-1}AS,$$

where $D$ is a diagonal matrix with diagonal entry $(D)_{ii} = \lambda_i$ and $\lambda_i$ is the eigenvalue of $A$ associated to the eigenvector $\vec{w}_i$: $A\vec{w}_i = \lambda_i \vec{w}_i$.

If $A$ is diagonalizable, and $k \geq 0$ is an integer, how can you compute $A^k$? Here’s how: $A$ diagonalizable implies that for some invertible matrix $S$, $D = S^{-1}AS$ is diagonal. We then have

$$D^k = (S^{-1}AS)^k = S^{-1}D^kS.$$ Moving the $S$’s to the left side, we obtain

$$SD^kS^{-1} = A^k.$$ So if you know $S$ and $S^{-1}$ (it is easy to compute $D^k$ if $D$ is diagonal), you can compute $A^k$.

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**Orthogonal matrices:** Their definition and basic properties:

- \( Q \in \text{Mat}_{n \times n}(\mathbb{R}) \) is orthogonal if and only if its rows and columns form orthonormal bases for \( \mathbb{R}^n \).
- If you rearrange the rows or columns of an orthogonal matrix, the resulting matrix is still orthogonal.
- If \( Q \) is orthogonal, then:
  - \( \forall \bar{x} \in \mathbb{R}^n, \|Q\bar{x}\| = \|\bar{x}\|. \) (Multiplication by \( Q \) preserves length)
  - \( \forall \bar{x}, \bar{y} \in \mathbb{R}^n, Q\bar{x} \cdot Q\bar{y} = \bar{x} \cdot \bar{y}. \) (Multiplication by \( Q \) preserves the angle between vectors.)
  - \( \det(Q) = \pm 1. \)
- \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \) is symmetric if and only if it is **orthogonally diagonalizable**. i.e., \( \exists Q, \text{ orthogonal}, \text{ such that } Q^{-1}AQ = D \) is diagonal.

§5.2: **Vector spaces:** The definition of vector space (a set \( V \) and a scalar field \( F \) together with an addition operation on \( V \) and a scalar multiplication operation); in particular, the ten vector space axioms: 2 closure axioms, 4 axioms for vector addition, 4 axioms for scalar multiplication. Examples of vector spaces: \( \text{Mat}_{m \times n}(\mathbb{R}), P_n \). Check whether a set \( V \) together with an addition and scalar multiplication is or is not a vector space.

§5.3: **Subspaces:** Determination of whether or not certain subsets of \( \text{Mat}_{m \times n}(\mathbb{R}) \) or \( P_n \) are subspaces.