1. **(15 points)** Find the determinant of

\[ A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 5 & 0 \end{pmatrix}. \]

Is \( A \) invertible? Why or why not?

You may use any method. This computation should not be long or complicated.

Expand across row 1:

\[
-2 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} = -2(-1) \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} 
\]

\[
= (2 - 4)(5 - 3) = -4.
\]
2. (a) (10 points) Let

\[
A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad B = \begin{pmatrix} d & 5e + 2d & f \\ a & 5b + 2a & c \\ g & 5h + 2g & i \end{pmatrix}.
\]

Suppose that \(\det(A) = 3\). What is \(\det(B)\)?

Hint: Consider row and column operations.

\[
\det(B) = \begin{vmatrix} d & 5e & f \\ a & 5b & c \\ g & 5h & i \end{vmatrix}. \quad \text{Now, do: } R_1 \leftrightarrow R_2 \\
C_2 \mapsto \frac{1}{5} C_2 \quad \text{to get } \det(B) = -5 \det(A) = -15.
\]

(b) (10 points) Suppose that \(A, B \in \text{Mat}_{4 \times 4}(\mathbb{R})\) are invertible matrices.

Suppose also that \(\det(A) = 3\) and \(\det(B) = 5\). What is \(\det(2A^{-3}B^T)\)?

\[
2^4 \cdot \frac{1}{3^3} \cdot 5 = \frac{80}{27}
\]

(c) (12 points) State whether the following are true (T) or false (F) in general.

- If \(A, B \in \text{Mat}_{n \times n}(\mathbb{R})\) are similar, then they have the same eigenvalues. \(\text{T}\)

- If \(A, B \in \text{Mat}_{n \times n}(\mathbb{R})\) are similar, then they have the same eigenvectors. \(\text{F}\)

- If \(A, B \in \text{Mat}_{n \times n}(\mathbb{R})\) have the same characteristic polynomial, then they are similar. \(\text{F}\)
3. (15 points) Find the eigenvalues of $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

The eigenvalues are integers. This should be a short computation.

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{vmatrix} = (3 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (3 - \lambda)((2 - \lambda)^2 - 1)$$

$$= (3 - \lambda)(4 - 4\lambda + \lambda^2 - 1) = (3 - \lambda)(\lambda^2 - 4\lambda + 3) = (3 - \lambda)(\lambda - 3)(\lambda - 1)$$

$\Rightarrow$ eigenvalues are $\lambda = 3$ (with alg. mult. 2)

$\lambda = 1$ (--- 1)
4. (15 points) Suppose you know that \( A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix} \) has eigenvalues \( \lambda = 3 \) (with algebraic multiplicity 2) and \( \lambda = 2 \) (with algebraic multiplicity 1).

Compute \( E_3 \), the eigenspace associated to \( \lambda = 3 \). Is \( A \) diagonalizable? Why or why not?

\[
E_3 = \text{null}(A - 3I) . \ (A - 3I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\begin{align*}
&x_1 = x_2 + x_3 \\
&x_2, x_3 \in \mathbb{R}, \ \text{mb.}
\end{align*}
\]

\[
E_3 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}
\]

\( \lambda = 3 \) has \( \text{alg. mult.} = 2 \) and \( \text{geom. mult.} = 2 \) \( \Rightarrow \) \( A \) is diagonalizable.

\( \lambda = 2 \) \( \Rightarrow \) \( A \) is diagonalizable.
5. Suppose that $A \in \text{Mat}_{3 \times 3}(\mathbb{R})$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 4$.

(a) \textbf{(10 points)} Suppose that \( \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \) is an eigenvector for $\lambda_1$, that \( \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \) is an eigenvector for $\lambda_2$, and that \( \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} \) is an eigenvector for $\lambda_3$.

Give a matrix $S$ which diagonalizes $A$ and the diagonal matrix $D$ obtained by diagonalizing $A$ with $S$. (Note that $A$ is diagonalizable since it has distinct eigenvalues.)

\[
S = \begin{pmatrix} 1 & 2 & -1 \\ 5 & 1 & 3 \\ 3 & 2 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\]

(c) \textbf{(6 points)} What are the eigenvalues of $A^2 + 3A + I$?

\[
\lambda_1 = 2^2 + 3 \cdot 2 + 1 = 11,
\lambda_2 = 3^2 + 3 \cdot 3 + 1 = 19,
\lambda_3 = 4^2 + 3 \cdot 4 + 1 = 29.
\]
6. (7 points) Suppose that $A, B \in \text{Mat}_{n \times n}(\mathbb{R})$ are non-singular matrices and that $\lambda$ is an eigenvalue of $AB$. Show that $\lambda$ is also an eigenvalue of $BA$.

Start by writing down what it means for $\lambda$ to be an eigenvalue of $AB$.

\[ \exists \, \vec{v} \in \mathbb{R}^n \text{ with } (AB) \vec{v} = \lambda \vec{v} \implies B(AB) \vec{v} = B(\lambda \vec{v}) = \lambda (B \vec{v}) \]

But $B(AB) \vec{v} = BA(B \vec{v})$. Hence, $BA(B \vec{v}) = \lambda (B \vec{v})$.

i.e., $\lambda$ is an eigenvalue of $BA$. A corresponding eigenvector is $B \vec{v}$. 