Math 241, Final Exam. 5/4/13. Name: ________________

• No notes, calculator, or text.
• There are 200 points total. Partial credit may be given.
• Circle or otherwise clearly identify your final answer.

1. **(15 points):** Find the equation of the plane that passes through the point $(1, 2, 3)$ and contains the line given by parametric equations $x = 3t$, $y = 1 + t$, $z = 2 - t$. State your equation in the form $ax + by + cz = d$.

**Solution:** The line lies in the plane and has direction vector $\vec{v} = \langle 3, 1, -1 \rangle$. To get a second vector parallel to the plane, we use the given point and the point $(0, 1, 2)$ to get $\vec{u} = \langle 1, 1, 1 \rangle$.

A normal vector to the plane, therefore, is

$$u \times v = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 1 & 1 \\
3 & 1 & -1
\end{vmatrix} = \vec{i}(-1-1) - \vec{j}(-1-3) + \vec{k}(1-3) = -2\vec{i} + 4\vec{j} - 2\vec{k} = -2(\vec{i} - 2\vec{j} + \vec{k}).$$

With normal vector $\vec{n} = \vec{i} - 2\vec{j} + \vec{k}$ and the point $(0, 1, 2)$, the plane equation is

$$1(x - 0) - 2(y - 1) + 1(z - 2) = 0 \iff x - 2y + 2 + z - 2 = x - 2y + z = 0.$$

2. **(15 points):** Suppose that $w = f(x, y, z)$, $y = g(s, t)$, and $z = h(t)$. Write down the form of the chain rule that you would use to compute $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

**Solution:** We have

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}.$$

3. **(15 points):** Find parametric equations for the line normal to the surface $\sin(xyz) = x + 2y + 3z$ at the point $(2, -1, 0)$.

**Solution:** Let $F(x, y, z) = \sin(xyz) - x - 2y - 3z$. Then the normal line has direction vector at $(2, 1, 0)$ given by

$$\nabla F(2, 1, 0) = \begin{vmatrix}
yz \cos(xyz) - 1 & xz \cos(xyz) - 2 & xy \cos(xyz) - 3
\end{vmatrix}_{(2, 1, 0)} = \langle -1, -2, -2 - 3 \rangle = \langle -1, -2, -5 \rangle = -\langle 1, 2, 5 \rangle.$$

For convenience, we use the parallel vector $\vec{v} = \langle 1, 2, 5 \rangle$. The parametric equations are

$$x = 2 + t, \quad y = 1 + 2t, \quad z = 5t; \quad t \in \mathbb{R}.$$
4. (15 points): Let $F(x, y, z) = xy + 2xz - y^2 + z^2$.

(a) (10 points): Find the directional derivative of $F(x, y, z)$ at the point $(1, -2, 1)$ in the direction of the vector $\vec{v} = (1, 1, 2)$.

Solution: We first normalize $\vec{v}$ to get $\vec{u} = \frac{1}{\| (1, 1, 2) \|} (1, 1, 2)$. Next, we compute

$$\nabla F(1, -2, 1) = \left. \langle y + 2z, x - 2y, 2x + 2z \rangle \right|_{(1, -2, 1)} = \langle 0, 5, 4 \rangle$$

We now have $D_{\vec{u}}F(1, -2, 1) = \nabla F(1, -2, 1) \cdot \vec{u} = \langle 0, 5, 4 \rangle \cdot (1/\sqrt{6}) (1, 1, 2) = 13/\sqrt{6}$.

(b) (5 points): Find the maximum rate of change of $F(x, y, z)$ at the point $(1, -2, 1)$.

Solution: From (a), we have $\nabla F(1, -2, 1) = \langle 0, 5, 4 \rangle$. The maximum rate of change is $\| \langle 0, 5, 4 \rangle \| = \sqrt{41}$.

5. (15 points): The function $f(x, y) = 2x^2y - 8xy + y^2 + 5$ has three critical points. Find the critical points and determine whether each is a local maximum, local minimum, saddle point, or none of these.

Solution: We find critical points by solving the system

$$f_x = 4xy - 8y = 0 \iff 4(x-2)y = 0, \quad f_y = 2x^2 - 8x + 2y = 0 \iff 2(x^2 - 4x + y) = 0.$$

The first equation gives $y = 0$ or $x = 2$. When $y = 0$, the second equation becomes $x^2 - 4x = x(x-4) = 0$, which gives the two points $(0, 0)$ and $(4, 0)$. When $x = 2$, the second equation gives $y = 4$, and hence the point $(2, 4)$. For the Second Derivative Test, we compute $f_{xx} = 4y$, $f_{yy} = 2$, and $f_{xy} = 4x - 8$. We now test the three points.

(a) $(0, 0)$. We compute $D = \begin{vmatrix} 0 & -8 \\ -8 & 0 \end{vmatrix} = -64 < 0$. It follows that $(0, 0)$ is a saddle point.

(b) $(4, 0)$. We compute $D = \begin{vmatrix} 8 & 0 \\ 8 & 2 \end{vmatrix} = -64 < 0$. It follows that $(4, 0)$ is a saddle point.

(c) $(2, 4)$. We compute $D = \begin{vmatrix} 16 & 0 \\ 0 & 2 \end{vmatrix} = 32 > 0$. Since $f_{xx}(2, 4) = 16 > 0$, we conclude that $(2, 4)$ is a local minimum.
6. **(15 points):** Use Lagrange multipliers to find the point(s) on the surface \( z = x^2 - 3y^2 - 1 \) closest to \((0, 0, 0)\). (You are asked to minimize the distance from a point \((x, y, z)\) on the surface to the point \((0, 0, 0)\).)

**Solution:** It suffices to minimize the square of the distance, \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraint \( g(x, y, z) = x^2 - 3y^2 - z = 1 \). We compute

\[
\nabla f = (2x, 2y, 2z) = 2\langle x, y, z \rangle, \quad \nabla g = (2x, -6y, -1).
\]

It follows that we must solve the system

\[
x = 2\lambda x, \quad y = -6\lambda y, \quad z = -\lambda, \quad x^2 - 3y^2 - z = 1.
\]

We first suppose that \( x \neq 0 \). The first equation gives \( \lambda = 1/2 \). Hence, we get \( z = -1/2 \) from the third equation. The second equation then becomes \( y = -6(1/2)y \), so \( y = 0 \). To get \( x \), we use the fourth equation: \( 1 = x^2 - 3y^2 - z = x^2 - (-1/2) = x^2 + 1/2 \). It follows that \( x = \pm 1/\sqrt{2} \). We obtain the points \((\pm 1/\sqrt{2}, 0, -1/2)\) when \( x \neq 0 \).

When \( x = 0 \), we consider the possibilities \( y = 0 \) and \( y \neq 0 \). If \( y \neq 0 \), the second equation implies that \( \lambda = -1/6 \). Therefore, the third equation gives \( z = 1/6 \). To get \( y \), we substitute in the fourth equation: \( 1 = x^2 - 3y^2 - z = -3y^2 - 1/6 < 0 \). We conclude that it is not possible for \( y \neq 0 \), so we must have \( y = 0 \). When \( x = 0 \) and \( y = 0 \), the fourth equation gives \( z = -1 \). We obtain a third point, \((0, 0, -1)\).

It remains to evaluate \( f(x, y, z) \) at the three points:

\[
f(\pm 1/\sqrt{2}, 0, -1/2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}; \quad f(0, 0, -1) = 1.
\]

We conclude that the points \((\pm 1/\sqrt{2}, 0, -1/2)\) are closest to \((0, 0, 0)\) on the surface \( g(x, y, z) = 1 \), at a distance of \( \sqrt{3/4} = \sqrt{3}/2 \).
7. **(15 points):** Evaluate the triple integral
\[
\int_0^\pi \int_0^2 \int_0^1 zx^2 \sin(xyz) \, dx \, dy \, dz.
\]
(The easiest way to do this integral is to arrange so that you do not have to use integration by parts.)

**Solution:** We compute
\[
\begin{align*}
\int_0^\pi \int_0^2 \int_0^1 zx^2 \sin(xyz) \, dx \, dy \, dz &= \int_0^\pi \int_0^2 \int_0^1 zx^2 \sin(xyz) \, dy \, dx \, dz \\
&= \int_0^\pi \int_0^1 \left( -zx^2 \cos(xyz) \right) \bigg|_0^2 \, dx \, dz \\
&= \int_0^\pi \int_0^1 \left( -x \cos(2xz) + x \right) \, dx \, dz \\
&= \int_0^\pi \left( -\frac{x \sin(2xz)}{2} + xz \right) \bigg|_0^1 \, dx \\
&= \int_0^\pi \left( -\frac{\sin(2\pi x)}{2} + \pi x \right) \, dx \\
&= \left( \frac{\cos(2\pi)}{4\pi} + \frac{\pi}{2} \right) - \left( \frac{\cos 0}{4\pi} + 0 \right) = \frac{\pi}{2}.
\end{align*}
\]

8. **(15 points):** Convert the double integral
\[
\int_0^6 \int_{\sqrt{6y-y^2}}^{\sqrt{6y+y^2}} \sqrt{1 + (x^2 + y^2)^{1/2}} \, dx \, dy
\]
to polar coordinates. (Consider drawing a quick sketch of the region in the \(xy\)-plane.)

**Do not evaluate.**

**Solution:** In order to sketch the region, we study the \(x\)-limits. We have
\[
x = \pm \sqrt{6y - y^2} \iff x^2 = 6y - y^2 \iff x^2 + y^2 - 6y + 9 = 9 \iff x^2 + (y - 3)^2 = 9.
\]

Hence, the region is a disk of radius 3 centered at (0, 3); it is tangent to the \(x\)-axis at the point (0, 0). It follows that \(0 \leq \theta \leq \pi\). For bounds on the polar variable \(r\), we convert \(x^2 + y^2 = 6y\) to \(r^2 = 6r \sin \theta\) to obtain \(r = 6 \sin \theta\). To conclude, we have
\[
\int_0^6 \int_{\sqrt{6y-y^2}}^{\sqrt{6y+y^2}} \sqrt{1 + (x^2 + y^2)^{1/2}} \, dx \, dy = \int_0^\pi \int_0^{6 \sin \theta} r \sqrt{1 + r} \, dr \, d\theta.
\]

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9. **(20 points):** Consider the triple integral

\[ \int_0^4 \int_0^{\sqrt{x}} \int_0^{4-2y} f(x, y, z) \, dz \, dy \, dx. \]

(Draw a quick sketch of the three-dimensional region of integration.)

(a) **(10 points):** Convert the triple integral to a triple integral in rectangular coordinates with \( dV = dx \, dz \, dy. \)

**Solution:** The region has four sides; three are planes, and the fourth is curved. The planes are \( z = 0 \) (xy-plane), \( x = 0 \) (yz-plane), and \( 2y + z = 4 \). The fourth side has an edge on the \( z \)-axis \( (0 \leq z \leq 4) \), an edge in the \( xy \)-plane \((y = \sqrt{x})\), and an edge where the plane \( 2y + z = 4 \) meets the parabolic cylinder \( y = \sqrt{x} \). We have

\[ \int_0^2 \int_0^{4-2y} \int_0^{\sqrt{x}} dx \, dz \, dy. \]

(b) **(10 points):** Convert the triple integral to a triple integral in rectangular coordinates with \( dV = dy \, dz \, dx \). (What is the equation for the boundary of the projection on the \( xz \)-plane?)

**Solution:** We note that the equation for the projection on the \( xz \)-plane arises from the intersection of \( 2y + z = 4 \) with \( y = \sqrt{x} \), which gives \( z = 4 - 2\sqrt{x} \) or \( x = (1/4)(z-4)^2 \). This is a parabola in \( xz \)-plane with vertex at \( x = 0, z = 4 \) which opens up in the positive \( x \)-direction. We have

\[ \int_0^4 \int_0^{4-2\sqrt{x}} \int_0^{\sqrt{x}} dy \, dz \, dx. \]

10. **(15 points):** Set up but **do not evaluate** a triple integral in spherical coordinates for the volume of the solid bounded above by the sphere \( x^2 + y^2 + z^2 = 4 \) and below by the cone \( 3z^2 = x^2 + y^2 \) with \( x, y, z \geq 0 \). (Draw a quick sketch of the solid.)

**Solution:** The solid is a cone (above the \( xy \)-plane) with a spherical cap. To compute bounds on the spherical variable \( \phi \), we see that \( 3z^2 = x^2 + y^2 \) gives \( z = \sqrt{3} \) or \( z = -\sqrt{3} \), so we have \( \tan \phi = \frac{\sqrt{x^2 + y^2}}{z} = \frac{\pm z \sqrt{3}}{z} = \pm \sqrt{3} \). Since the solid lies above the \( xy \)-plane, we have \( \tan \phi = \sqrt{3} \) with \( 0 \leq \theta \leq \pi \). Hence, we get \( \phi = \pi/3 \). The integral giving the volume in spherical coordinates is then

\[ \text{Volume} = \int_0^{\pi/2} \int_0^{\pi/3} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \]
11. (15 points): Evaluate the double integral \( \int \int_R \cos \left( \frac{x - y}{x + y} \right) \, dA \), where \( R \) is the trapezoidal region in the first quadrant with vertices \((1, 0), (3, 0), (0, 1)\), and \((0, 3)\).

(Use a suitable change of variables from \((x, y)\) to \((u, v)\) to evaluate a double integral over a region \(S\) in the \(uv\)-plane. To get limits on the new variables \(u\) and \(v\), it may be useful to draw \(S\); you can do this by plotting a few well-chosen points. What are the equations for the boundary of \(S\)?)

**Solution:** We use the change of variables \(u(x, y) = x - y, \ v(x, y) = x + y\). Then we have \(J^{-1} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2\), so \(J = 1/2\). We observe that points \((x, y)\) in \(R\) map to points \((u(x, y), v(x, y))\) in \(S\). Since \(R\) is a trapezoid and \(u(x, y), v(x, y)\) are linear, the region \(S\) must be a quadrilateral (4 sides, 4 vertices). The vertices of \(S\) are

\[
(u(0, 1), v(0, 1)) = (-1, 1), \ (u(0, 3), v(0, 3)) = (-3, 3), \\
(u(1, 0), v(0, 1)) = (1, 1), \ (u(3, 0), v(3, 0)) = (3, 3).
\]

Hence, \(S\) is a trapezoid with sides \(v = 1, v = 3\) (parallel), and \(u = v, u = -v\). We now compute

\[
\int \int_R \cos \left( \frac{x - y}{x + y} \right) \, dA = \frac{1}{2} \int_1^3 \int_{-v}^v \cos \left( \frac{u}{v} \right) \, du \, dv = \frac{1}{2} \int_1^3 \left( v \sin \left( \frac{u}{v} \right) \right)_{-v}^v \, dv \\
= \frac{1}{2} \int_1^3 (v \sin 1 - (v \sin(-1))) \, dv = \frac{1}{2} \int_1^3 2v \sin 1 \, dv = \sin 1 \int_1^3 v \, dv \\
= (\sin 1) \left( \frac{3^2}{2} - \frac{1}{2} \right) = 4 \sin 1.
\]
12. Some exams had problem (a); some had problem (b).

(a) §16.3, #15: Let \( \vec{F}(x, y, z) = yz\vec{i} + xz\vec{j} + (xy + 2z)\vec{k} \), and let \( C \) be the line segment from \((1, 0, -2)\) to \((4, 6, 3)\).

(b) §16.3, #18: Let \( \vec{F}(x, y, z) = e^y\vec{i} + xe^y\vec{j} + (z + 1)e^z\vec{k} \), and let \( C \) be the curve given by \( \vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \) \( 0 \leq t \leq 1. \)

i. **(10 points)** Find a function \( f \) such that \( \vec{F} = \nabla f. \)

**Solution:** For (a), we seek \( f(x, y, z) \) with \( \partial f/\partial x = yz, \partial f/\partial y = xz, \) and \( \partial f/\partial z = xy + 2z. \) First, we compute \( f(x, y, z) = \int \frac{\partial f}{\partial x} \, dx + g(y, z) = \int yz \, dx + g(y, z) = xyz + g(y, z). \) Hence, we have \( \partial f/\partial y = xz + g_y(y, z) = xz. \) It follows that \( g_y(y, z) = 0. \) Therefore, we obtain \( g(y, z) = \int g_y(y, z) \, dy + h(z) = h(z), \) which gives \( f(x, y, z) = xyz + h(z). \) To conclude, we compute \( \partial f/\partial z = xy + h'(z) = xy + 2z. \) We obtain

\[
h(z) = \int h'(z) \, dz = \int 2z \, dz = z^2 + K
\]

We now have \( f(x, y, z) = xyz + z^2 + K. \)

For (b), we seek \( f(x, y, z) \) with \( \partial f/\partial x = e^y, \partial f/\partial y = xe^y, \) and \( \partial f/\partial z = (z + 1)e^z. \) First, we compute \( f(x, y, z) = \int \frac{\partial f}{\partial x} \, dx + g(y, z) = \int e^y \, dx + g(y, z) = xe^y + g(y, z). \) Hence, we have \( \partial f/\partial y = xe^y + g_y(y, z) = xe^y. \) It follows that \( g_y(y, z) = 0. \) Therefore, we obtain \( g(y, z) = \int g_y(y, z) \, dy + h(z) = h(z), \) which gives \( f(x, y, z) = xe^y + h(z). \) To conclude, we compute \( \partial f/\partial z = h'(z) = (z + 1)e^z. \) Using integration by parts with \( u = z + 1 \) and \( dv = e^z \, dz, \) we obtain

\[
h(z) = \int h'(z) \, dz = \int (z+1)e^z \, dz = (z+1)e^z - \int e^z \, dz = (z+1)e^z - e^z = ze^z + K.
\]

We now have \( f(x, y, z) = xe^y + ze^z + K. \)

ii. **(5 points)** Use part (a) to evaluate \( \int_C \vec{F} \cdot d\vec{r} \) along the given curve \( C. \)

**Solution:** For (a), the Fundamental Theorem of Calculus for line integrals gives

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77.
\]

For (b), the curve \( C \) has initial point \( \vec{r}(0) = \langle 0, 0, 0 \rangle \) and terminal point \( \vec{r}(t) = \langle 1, 1, 1 \rangle. \) The Fundamental Theorem of Calculus for line integrals gives

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0) = 2e.
\]
13. §16.4, #12 (15 points): Let \( \vec{F}(x,y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle \), and let \( C \) be the triangle from \((0,0)\) to \((2,6)\) to \((2,0)\) to \((0,0)\). Use Green’s Theorem to evaluate \( \int_C \vec{F} \cdot d\vec{r} \).

(\text{Check the orientation of the curve before applying the theorem.)}

\textbf{Solution:} We have \( P(x,y) = y^2 \cos x \), \( Q(x,y) = x^2 + 2y \sin x \). We note that \( C \) is a simple closed curve enclosing the triangle \( D \) in the first quadrant with sides given by \( y = 0 \), \( x = 2 \), and \( y = 3x \). Since \( C \) is oriented negatively (in the clockwise direction), we use Green’s Theorem as follows:

\[
\int_C \vec{F} \cdot d\vec{r} = \int_{C^-} P(x,y) \, dx + Q(x,y) \, dy = -\int_{C^+} P(x,y) \, dx + Q(x,y) \, dy
\]

\[
= -\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = -\int_0^2 \int_0^{3x} ((2x + 2y \cos x) - 2y \cos x) \, dy \, dx
\]

\[
= -\int_0^2 \int_0^{3x} 2x \, dy \, dx = -2 \int_0^2 3x^2 \, dx = -6 \cdot \frac{8}{3} = -16.
\]