Math 241, Final Exam. 4/28/12. Name: ____________________

- No notes, calculator, or text.
- There are 200 points total. Partial credit may be given.
- Write your full name in the upper right corner of page 1.
- Do problem 1 on p.1, problem 2 on p.2,... (or at least present your solutions in numerical order).
- Circle or otherwise clearly identify your final answer.

1. (10 points): Suppose that \( \vec{u} \) and \( \vec{v} \) are vectors in \( \mathbb{R}^3 \) with \( ||\vec{u}|| = 3, \ ||\vec{v}|| = 5 \), and \( \vec{u} \cdot \vec{v} = 9 \).

   (a) (5 points): What is the cosine of \( \theta \), the angle between \( \vec{u} \) and \( \vec{v} \)?

   **Solution:** We have
   \[
   \cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} = \frac{9}{15} = \frac{3}{5}.
   \]

   (b) (5 points): What is \( ||\vec{u} \times \vec{v}|| \)? You may use the result of part (a), if you choose.

   **Solution:** We have
   \[
   ||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin \theta = ||\vec{u}|| ||\vec{v}|| \sqrt{1 - \cos^2 \theta} = (3)(5) \sqrt{1 - (3/5)^2} = (3)(5)(4/5) = 12
   \]

2. (15 points): Lines.

   (a) (10 points): Find parametric equations for the line \( L \) through the points \((1, 0, 1)\) and \((4, -2, 2)\).

   **Solution:** The direction vector is \( \vec{v} = (4, -2, 2) - (1, 0, 1) = (3, -2, 1) \). Hence the line has parametric equations
   \[
   x = 1 + 3t, \quad y = -2t, \quad z = 1 + t \quad \text{for } t \in \mathbb{R}.
   \]

   (b) (5 points): The line \( L \) intersects the plane \( M : x + y + z = 6 \) in a point \( P \). Find \( P \). You may use the result of part (a), if you choose.

   **Solution:** We have
   \[
   x + y + z = (1 + 3t) + (-2t) + (1 + t) = 2 + 2t = 6 \quad \Rightarrow \quad t = 3.
   \]
   Therefore, the point \( P \) is \((7, -6, 4)\).
3. (15 points): Planes. Find the equation of the plane that contains the line

\[ \frac{x - 1}{2} = \frac{y - 2}{2} = \frac{z - 1}{3} \]

and is perpendicular to the plane \(4x - y + 2z = 7\). Give your answer in the form \(ax + by + cz = d\). (If you prefer to think of the line in parametric form, set each fraction in the display equal to a parameter \(t \in \mathbb{R}\), and solve for \(x, y, z\).)

**Solution:** The line has direction vector \(\vec{v} = \langle 2, 2, 3 \rangle\). The plane has normal vector \(\vec{n} = \langle 4, -1, 2 \rangle\). The desired plane must have normal vector orthogonal to \(\vec{v}\) and \(\vec{n}\). We compute

\[
\vec{v} \times \vec{n} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
2 & 2 & 3 \\
4 & -1 & 2
\end{vmatrix} = 7\vec{i} + 8\vec{j} - 10\vec{k}.
\]

To conclude, we note from the equation of the line that \((1, 2, 1)\) is a point in the desired plane. Therefore, this plane has equation

\[7(x - 1) + 8(y - 2) - 10(z - 1) = 0 \iff 7x + 8y - 10z = 13.\]

4. (5 points): Suppose you know that \(f(x, y, z)\) has \(f_{xz} = x\sqrt{2y - z}\). Find \(f_{yzx}\).

**Solution:** Clairaut’s Theorem implies that \(f_{xz} = f_{yzx}\). It follows that

\[f_{yzx} = \frac{\partial}{\partial y} \left( x(2y - z)^{1/2} \right) = x(2(1/2)(2y - z)^{-1/2}).\]

5. (10 points): Tangent plane. Find an equation of the form \(ax + by + cz = d\) for the plane tangent to the surface \(F(x, y, z) = x^2 + 2y^2 - 3z^2 = 3\) at the point \((2, -1, 1)\).

**Solution:** We compute \(F_x = 2x, F_y = 4y,\) and \(F_z = -6z\). Hence, at \((2, -1, 1)\), we have \(F_x = 4, F_y = -4, F_z = -6\). The tangent plane has equation

\[4(x - 2) - 4(y + 1) - 6(z - 1) = 0 \iff 4x - 4y - 6z = 6 \iff 2x - 2y - 3z = 3.\]
6. **(15 points): Chain rule.** Suppose that \(z = f(x, y)\), and that \(x = g(u, v, w)\) and \(y = h(v)\).

(a) **(9 points):** Write down the form of the chain rule that you would use to compute \(\frac{\partial z}{\partial v}\).

**Solution:** We have
\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.
\]

(b) **(6 points):** Suppose you know that \(g(1, 4, 3) = 2\) and \(h(4) = 7\) so that \(f(g(1, 4, 3), h(4)) = (2, 7)\). Suppose you also know that

i. \(g_v(1, 4, 3) = -1\),

ii. \(\left.\frac{dh}{dv}\right|_{v=4} = h'(4) = 3\),

iii. \(f_x(2, 7) = 6\) and \(f_y(2, 7) = 1\).

Compute the value of \(\frac{\partial z}{\partial v}\) at \((1, 4, 3)\). You may use the result of part (a), if you choose. (It may be helpful to think about how to write \(g_v, h', f_x,\) and \(f_y\) in “\(\partial\) notation”.)

**Solution:** We have
\[
\frac{\partial z}{\partial v} = (6)(-1) + (1)(3) = -3.
\]

7. **(15 points): Directional derivative.** Let \(F(x, y, z) = xy + yz + zx\).

(a) **(10 points):** Find the directional derivative of \(F(x, y, z) = xy + yz + zx\) at \((1, -1, 3)\) in the direction of \((2, 4, 5)\).

**Solution:** The gradient is \(\nabla F = (y + z, x + z, x + y)\); evaluated at \((1, -1, 3)\), we obtain \(\nabla F = (2, 4, 0)\). The required direction is \((2, 4, 5) - (1, -1, 3) = (1, 5, 2)\); a unit vector in this direction is
\[
\vec{u} = \frac{(1, 5, 2)}{\sqrt{1 + 25 + 4}} = \left\langle \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, 0 \right\rangle.
\]

To conclude, we have
\[
D_{\vec{u}}F(1, -1, 3) = \nabla F \cdot \vec{u} = \frac{22}{\sqrt{30}}.
\]

(b) **(5 points):** Give a unit vector that points in the direction of fastest increase of \(F\). You may use the result of part (a), if you choose.

**Solution:** The required unit vector is \(\nabla F / \|\nabla F\| = \left\langle \frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}}, 0 \right\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle\).
8. **(20 points): Extrema.** Let \( R \) be the rectangle with vertices \((0, 0), (4, 0), (0, 5), (4, 5)\), and let \( f(x, y) = 4x + 6y - x^2 - y^2 \).

(a) **(10 points):** Find all local maxima, minima, and saddle points of \( f(x, y) \) in the interior of \( R \).

**Solution:** We compute \( f_x = 4 - 2x \) and \( f_y = 6 - 2y \). Solving \( f_x = 0 \) and \( f_y = 0 \) together gives \( x = 2, \ y = 3 \). Furthermore, we observe that \( f_{xx} = -2 < 0, f_{yy} = -2, \) and \( f_{xy} = 0 \). Therefore, \( D = (-2)^2 > 0 \), and we conclude by the second derivative test that \( f(x, y) \) has a local maximum at \((2, 3)\).

(b) **(10 points):** Find the absolute maximum and minimum of \( f(x, y) \) in \( R \). Justify your conclusions; be complete. Useful information (which you may assume):

i. \( R \) is bounded by the lines: \( L_1 : x = 0, \ L_2 : y = 5, \ L_3 : x = 4, \ L_4 : y = 0 \).

ii. Values of \( f \) at the corners of \( R \):
   \[ f(0, 0) = 0, \ f(4, 0) = 0, \ f(0, 5) = 5, \ f(4, 5) = 5. \]

**Solution:** It suffices to evaluate \( f \) at the critical point from (a), \((2, 3)\), and at candidate points on the boundary of \( R \). We analyze the boundary:

\( L_1 : x = 0. \) We analyze \( f(0, y) = 6y - y^2 \) for \( 0 \leq y \leq 5 \).
   We have \( f'(0, y) = 6 - 2y = 0 \iff y = 3 \). Candidate for extremum: \((0, 3)\).

\( L_2 : y = 5. \) We analyze \( f(x, 5) = 4x - x^2 + 5 \) for \( 0 \leq x \leq 4 \).
   We have \( f'(x, 5) = 4 - 2x = 0 \iff x = 2 \). Candidate for extremum: \((2, 5)\).

\( L_3 : x = 4. \) We analyze \( f(4, y) = 6y - y^2 \) for \( 0 \leq y \leq 5 \).
   We have \( f'(4, y) = 6 - 2y = 0 \iff y = 3 \). Candidate for extremum: \((4, 3)\).

\( L_4 : y = 0. \) We analyze \( f(x, 0) = 4x - x^2 \) for \( 0 \leq x \leq 4 \).
   We have \( f'(x, 0) = 4 - 2x = 0 \iff x = 2 \). Candidate for extremum: \((2, 0)\).

We will compare the values of \( f \) at these points with the values at the four corners, already given. We have
\[ f(2, 3) = 15, \ f(0, 3) = 9, \ f(2, 5) = 9, \ f(4, 3) = 9, \ f(2, 0) = 4. \]

It follows that the absolute maximum is 15, located at the interior critical point \((2, 3)\); the absolute minimum is 0, and occurs at both corners \((0, 0)\) and \((4, 0)\).
9. **(15 points): Lagrange multipliers.** Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to the constraint $g(x, y) = x^2 + 4y^2 - 4 = 0$.

**Solution:** We compute gradients:

$$\nabla f = (-2x, 2y), \quad \nabla g = (2x, 8y)$$

The vector equation $\nabla f = \lambda \nabla g$ is equivalent to the system

$$-2x = 2\lambda x, \quad 2y = 8\lambda y \iff -x = \lambda x, \quad y = 4\lambda y.$$ 

If $x = 0$, we solve $g(0, y) = 0$ for $y$ to obtain $y = \pm 1$, giving points $(0, \pm 1)$. If $x \neq 0$, then $\lambda = -1$, and we have $y = -4y$, which gives $y = 0$. We solve $g(x, 0) = 0$ to obtain $x = \pm 2$, giving points $(\pm 2, 0)$. It remains to test values of $f$ at the given points:

$$f(\pm 2, 0) = -4, \quad f(0, \pm 1) = 1.$$ 

Subject to $g(x, y) = 0$, it follows that $f$ has maximum value 1 at $(0, \pm 1)$ and minimum value $-4$ at $(2, 0)$.

10. **(17 points):** Evaluate the double integral

$$\int_0^4 \int_{\sqrt{y}}^2 \sin x \frac{x}{x^2} \, dx \, dy.$$ 

Your answer should be a number.

**Solution:** Reversing the order, the integral becomes

$$\int_0^2 \int_0^{x^2} \frac{x}{x^2} \sin x \, dy \, dx = \int_0^2 \left[ \frac{x}{x^2} \sin x \right]_{y=0}^{x^2} \, dx = \int_0^2 \sin x \, dx = -\cos x \bigg|_{x=0}^{x=2} = -(\cos 2 - 1) = 1 - \cos 2.$$ 

11. **(15 points): Rectangular coordinates.** Express the iterated integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx$$

as an iterated integral with $dV = dz \, dx \, dy$. (Draw a quick sketch. What is the shadow (projection) of $E$, the three-dimensional region of integration, on the $xy$-plane?)

**Solution:** With $dV = dz \, dx \, dy$, the integral becomes

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z) \, dz \, dx \, dy.$$
12. **(15 points): Cylindrical coordinates.** Let $E$ be the solid region that lies within the cylinder $x^2 + y^2 = 4$, above the $xy$-plane ($z = 0$), and below $z = 9x^2 + 9y^2$. Express the integral $\iiint_{E} x \, dV$ as an iterated integral in *cylindrical coordinates*. 

**Solution:** In cylindrical coordinates, the integral becomes

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{9r^2} r^2 \cos \theta \, dz \, dr \, d\theta.$$ 

13. **(15 points): Spherical coordinates.** Convert the triple integral

$$\int_{0}^{3} \int_{\sqrt{9-x^2}}^{\sqrt{9-x^2-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} yz \, dz \, dy \, dx$$

to *spherical coordinates*. (Draw a quick sketch of $E$, the three-dimensional region of integration.) **Do not evaluate.**

**Solution:** In spherical coordinates, the integral becomes

$$\int_{0}^{\pi/2} \int_{0}^{\pi} \int_{0}^{3} \rho^4 \sin^2 \phi \cos \phi \cos \theta \, d\rho \, d\phi \, d\theta.$$ 

14. **(18 points):** Compute the double integral

$$\iint_{R} (2x - y) \cos(x - 3y) \, dA$$

over the parallelogram $R$ with sides given by equations

$$2x - y = 1, \ 2x - y = 3; \ x - 3y = 0, \ x - 3y = 2.$$ 

Use a suitable change of variable. Your answer should be a number.

**Solution:** We let $u = 2x - y$ and $v = x - 3y$. It follows that $1 \leq u \leq 3$ and $0 \leq v \leq 2$. We compute

$$\frac{\partial(u, v)}{\partial(x, y)} = \left| \begin{array}{cc} 2 & -1 \\ 1 & -3 \end{array} \right| = -6 - (1) = -5 \quad \implies \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{5}.$$ 

To conclude, we have

$$\iint_{R} (2x - y) \cos(x - 3y) \, dA = \int_{1}^{3} \int_{0}^{2} u \cos v \left| -\frac{1}{5} \right| \, dv \, du = \frac{1}{5} \left( \int_{1}^{3} u \, du \right) \left( \int_{0}^{2} \cos v \, dv \right)$$

$$= \frac{1}{5} \left( \frac{9}{2} - \frac{1}{2} \right) \sin 2 = \frac{4}{5} \sin 2.$$