Math 241, Final Exam. 12/8/11. Name: ________________

• No notes, calculator, or text.
• There are 200 points total. Partial credit may be given.
• Write your full name in the upper right corner of page 1.
• Do problem 1 on p.1, problem 2 on p.2,... (or at least present your solutions in numerical order).
• Circle or otherwise clearly identify your final answer.

1. (5 points): Give the cosine of the angle between the vectors

\[ \vec{u} = \vec{i} - \vec{j} + 2\vec{k}, \quad \vec{v} = 2\vec{i} - \vec{j} + \vec{k}. \]

2. (15 points): Lines. Find parametric equations for the line \( L \) through the point \((1, 2, 3)\) that is parallel to the plane \( Q : x + y + z = 2 \) and perpendicular to the line

\[ L_1 : x = 1 + 2t; \quad y = 1 - 2t; \quad z = t. \]

3. (15 points): Planes. Find the equation of the plane that passes through the point \((1, 2, 3)\) and contains the line

\[ L : x = 3t; \quad y = 1 + t; \quad z = 2 - t, \quad t \in \mathbb{R}. \]

4. (15 points): Tangent plane. Find the equation of the plane tangent to \( z = x^2y + 2xy^3 \) at the point \((1, 1, 3)\).

5. (15 points): Chain rule.
   
   (a) (10 points): Let \( z = e^{xy^2}, \ x = t \cos t, \ y = t \sin t \). Compute \( \frac{dz}{dt} \). You may leave your answer in the variables \( x, y, \) and \( t \).

   (b) (5 points): Suppose that

\[ z = F(x, y), \quad x = G(u, v), \quad u = H(t), \quad v = I(t). \]

   Write down the form of the chain rule that you would use to compute \( \frac{dz}{dt} \). (Use a tree diagram to show variable dependencies.)

6. (15 points): Gradient and directional derivative. Let \( F(x, y, z) = 2x^3y - 3y^2z \).
   
   (a) (10 points): Compute the directional derivative of \( F \) at \((1, -1, 1)\) in the direction of the vector \( \vec{u} = (2, 3, 6) \). Is \( F \) increasing or decreasing at this point?

   (b) (5 points): In what direction from \((1, -1, 1)\) is the directional derivative of \( F \) a maximum? Your answer need not be a unit vector.
7. (15 points): Local extrema. Let \( f(x, y) = x^3 + y^3 - 3x - 12y + 20 \). Find the point(s) \((x, y)\) at which \( f(x, y) \) has a local maximum, minimum, or saddle.

8. (15 points): Absolute extrema. Find the absolute maximum of \( f(x, y) = 2x + 3y + 1 \) on the triangular region \( D \) in the first quadrant bounded by:

\[ L_1 : x = 0 \text{ (y-axis)}; \quad L_2 : y = 0 \text{ (x-axis)}; \quad L_3 : x + y = 2. \]

Clearly explain your answer.

9. (15 points): Lagrange multipliers. Use Lagrange multipliers to find the point(s) \((x, y, z)\) with \( x, y, z \geq 0 \) which maximize \( f(x, y, z) = xyz^2 \) subject to the constraint \( x + 2y + 2z = 6 \).

10. (15 points): Polar coordinates. Set up an integral in polar coordinates for the volume of the solid \( E \) that lies below the plane \( z = 6 \) and above the paraboloid \( z = 3x^2 + 3y^2 \) in the first octant. (How do the plane and paraboloid intersect?)

11. (15 points): Rectangular coordinates. Suppose that \( E \) is the tetrahedron (a polyhedron with four vertices and four triangular faces, three of which meet at each vertex) bounded by the planes

\[ x = y, \quad x + y = 4, \quad y = z, \quad z = 0 \text{ (xy-plane)}. \]

Express the integral \( \iiint_E dV \) as an iterated integral with \( dV = dy \, dx \, dz \). Do not evaluate.

(Draw a quick sketch. What is the shadow (projection) of \( E \) on the \( xz \)-plane?)

12. (15 points): Cylindrical coordinates. Convert the integral

\[ \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} y\sqrt{x^2+y^2} \, dz \, dy \, dx \]

from rectangular to cylindrical coordinates. Do not evaluate.

13. (15 points): Spherical coordinates. Use spherical coordinates to evaluate the triple integral

\[ \iiint_E xy \, dV \]

where \( E \) is the solid region that lies within the sphere \( x^2 + y^2 + z^2 = 4 \) and below the cone \( z = \sqrt{x^2 + y^2} \) in the first octant.

14. (15 points): Change of variable in a double integral. Compute the double integral

\[ \iint_R \frac{x-y}{x+y} \, dA \]

over the square \( R \) with vertices \( \{(0, 2), (1, 1), (2, 2), (1, 3)\} \). Use a suitable change of variable.