
Name: _______________________

- Read problems carefully. Show all work.
- No notes.
- The exam is approximately 15 percent of the total grade.
- There are 100 points total. Partial credit may be given.
1. (10 points) Find an equation of the tangent plane to

$$ F(x, y, z) = xyz + 2x + 3y + 3z + 2 = 0 $$

at \((1, 2, -2)\).

$$ \nabla F = (yz+2) \hat{i} + (xz+3) \hat{j} + (xy+3) \hat{k} $$

$$ \nabla F(1, 2, -2) = (-4+2) \hat{i} + (-2+3) \hat{j} + (2+3) \hat{k} = -2 \hat{i} + \hat{j} + 5 \hat{k}. $$

Tangent plane:

$$ -2(x-1) + (y-2) + 5(z+2) = 0 $$

$$ \iff -2x + y + 5z + 10 = 0 $$

$$ \iff 2x - y - 5z = 10 $$

2. (20 points) Locate all relative maxima, minima, and saddle points for

\[ f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4. \]

\[ \frac{\partial f}{\partial x} = 2x - y + 2 = 0 \implies y = 2x + 2 \]

\[ \frac{\partial f}{\partial y} = -x + 2y + 2 = 0 \implies -x + 2(2x + 2) + 2 = 3x + 6 = 0 \implies x = -2 \]

\[ y = 2(-2) + 2 = -2 \implies \text{c. p. at } (-2, -2). \]

\[ \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = -1. \]

\[ D = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 2 = 2 > 0. \]

\[ \frac{\partial^2 f}{\partial x^2} > 0 \implies \text{rel. min. at } (-2, -2). \]
3. **(20 points)** Use Lagrange multipliers to find the point(s) \((x, y, z)\) with \(x, y, z \geq 0\) which maximize \(f(x, y, z) = xy^2z^2\) subject to the constraint \(g(x, y, z) = x + y + z - 5 = 0\).

\[
\nabla f = y^2z^2\hat{i} + 2xyz^2\hat{j} + 2xyz\hat{k}
\]

\[
\nabla g = \hat{i} + \hat{j} + \hat{k}
\]

\[
\nabla f = \lambda \nabla g \iff \begin{cases} \ n
\end{cases}
\]

\[\begin{align*}
\lambda y^2z^2 &= \lambda \\
2\lambda xz^2 &= \lambda \\
2\lambda xy^2 &= \lambda
\end{align*}\]

\[\begin{align*}
\lambda y^2z^2 &= 2\lambda xz^2 \\
\lambda xy^2 &= 2\lambda xy^2
\end{align*}\]

\[
\begin{align*}
\lambda &= 2x \\
y &= 2x \\
2xz &= \lambda
\end{align*}\]

Substitute \(\lambda\) in \(g\).

\[
\begin{align*}
g(x, y, z) &= x + (2x) + (2x) - 5 = 0 \\
\lambda &= 2x \\
\lambda &= 2x
\end{align*}\]

\[
\begin{align*}
\hat{\lambda} &= \boxed{1, 2, 2}
\end{align*}\]
4. **(10 points)** Let $R$ be the region in the $xy$-plane enclosed by

\[ y = x^2, \quad y = \frac{1}{x}, \quad \text{and} \quad y = 4. \]

Set up, but do not evaluate, an iterated integral (or sum of iterated integrals) which gives the area enclosed by the region $R$.

You may view the region $R$ as a type I region (integrate with respect to $y$ first) or type II region (integrate with respect to $x$ first).

**Type I:**

\[
\iint_{R_1} dA + \iint_{R_2} dA = \int_{1/4}^{1} \int_{1/x}^{4} dy \, dx + \int_{1}^{2} \int_{x^2}^{4} dy \, dx.
\]

**Type II:**

\[
\iint_{R} dA = \int_{1}^{4} \int_{\sqrt{y}}^{1} dx \, dy.
\]
5. (15 points) Consider the iterated integral

\[ \int_{0}^{4} \int_{\sqrt{y}}^{2} \sqrt{x^3 + 1} \, dx \, dy. \]

(a) (8 points) Express the iterated integral as an equivalent iterated integral with the order of integration reversed. You may want to first sketch the region \( R \) over which the integral is taken.

(b) (7 points) Evaluate the iterated integral that you obtained in part (a).

\[
\begin{align*}
= & \quad \int_{0}^{2} x^2 \sqrt{x^3 + 1} \, dx = \frac{1}{3} \int_{1}^{9} u^{\frac{1}{2}} \, du = \left( \frac{1}{3} \right) \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \bigg|_{u=1}^{9} \\
\text{(}u = x^3 + 1 \Rightarrow du = 3x^2 \, dx\text{)}
\end{align*}
\]

\[
\begin{align*}
= & \quad \frac{2}{9} (2^7 - 1) \\
= & \quad \frac{\sqrt{82}}{9}
\end{align*}
\]
6. (10 points) Set up, but do not evaluate an iterated integral in rectangular coordinates which gives the volume of the solid enclosed by

- \( z = 0 \) (the \( xy \)-plane),
- the plane \( y = z \),
- and the cylinder \( x^2 + y^2 = 4 \).

It is a good idea (but not essential) to first sketch the solid.

\[
V = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} y \, dy \, dx = 2 \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} y \, dy \, dx.
\]
7. (15 points) Consider the iterated integral

\[ \int_0^3 \int_0^{\sqrt{9-z^2}} e^{x^2+y^2} \, dy \, dx. \]

(a) (8 points) Express the integral as an equivalent integral in polar coordinates. You may want to first sketch the region over which the integral is taken.

\[ \int_0^{\pi/2} \int_0^3 e^{r^2} \, r \, dr \, d\theta \]

(b) (7 points) Evaluate the iterated integral that you obtained in part (a).

\[
\text{Let } u = r^2, \quad du = 2r \, dr \quad \Rightarrow \quad \frac{1}{2} \, du = r \, dr.
\]

\[
\frac{1}{2} \int_0^{\pi/2} \int_0^3 e^u \, du \, d\theta = \frac{1}{2} \int_0^{\pi/2} (e^q - 1) \, d\theta = \frac{1}{2} (e^q - 1) \frac{\pi}{2}
\]

\[
= \frac{\pi}{4} (e^q - 1)
\]