Math 241, Exam 3 Information.
11/21/11, LC 121, 11:15 - 12:05.

Exam 3 will be based on:

- Sections 15.2 - 15.4, 15.6 - 15.8.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/~boylan/SCCourses/241Fa11/241.html)
  
  **At minimum, you need to understand how to do the homework problems.**
- Lecture notes: 10/19- 11/16.

**Topic List (not necessarily comprehensive):**

You will need to know: theorems, results, and definitions from class.

**§15.2: Iterated integrals.**

**Definition:** Let $a \leq x \leq b$, $c \leq y \leq d$, and suppose that $f(x, y)$ is continuous on the rectangle $[a, b] \times [c, d]$. Then we have

$$\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy;$$

the variable $y$ is held constant in the inner integral on the right. We also have

$$\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx;$$

the variable $x$ is held constant in the inner integral on the right.

**Fubini’s Theorem:** Suppose that $f(x, y)$ is continuous on $D = [a, b] \times [c, d]$. Then we have

$$\int_a^b \int_c^d f(x, y) \, dy \, dx = \iint_D f(x, y) \, dA = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy.$$

**Note:** Switching the order of integration is only this easy because we are integrating over rectangles. More care must be taken over general regions.

**Fact:** Suppose that $f(x, y) = g(x)h(y)$ ($f(x, y)$ is “separable”), and that it is continuous on $D = [a, b] \times [c, d]$. Then we have

$$\iint_D f(x, y) \, dA = \left( \int_c^d h(y) \, dy \right) \left( \int_a^b g(x) \, dx \right).$$
**Fact:** The volume under \( z = f(x, y) \geq 0 \) and above the \( xy \)-plane over \( D = [a, b] \times [c, d] \) is

\[
\int\int_{D} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx.
\]

§15.3: **Double integrals over general regions.**

Suppose that \( f(x, y) \) is continuous on \( D \subseteq \mathbb{R}^2 \).

- **Type I:** A region \( D \) is of type I if and only if
  \[
  D = \{(x, y) : a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}.
  \]
  We have
  \[
  \int\int_{D} f(x, y) \, dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
  \]

- **Type II:** A region \( D \) is of type II if and only if
  \[
  D = \{(x, y) : h_1(y) \leq x \leq h_2(y), \ c \leq y \leq d\}.
  \]
  We have
  \[
  \int\int_{D} f(x, y) \, dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
  \]

**Facts:** Let \( D \subseteq \mathbb{R}^2 \) be a type I or type II region.

- **Volume:** Let \( f(x, y) \) be continuous on \( D \). Then the volume of the region above \( D \) and below \( z = f(x, y) \geq 0 \) is
  \[
  \int\int_{D} f(x, y) \, dA.
  \]

- **Area:** The area of \( D \) is
  \[
  \int\int_{D} \, dA.
  \]
§15.4: Double integrals in polar coordinates.

**Polar coordinates:** \( P = (r, \theta) \).
- \( r \geq 0 \): the distance from the origin (pole), \( O \), to the point \( P \).
- \( 0 \leq \theta \leq 2\pi \): the angle, measured counter-clockwise, that the ray \( OP \) makes with the positive \( x \)-axis.

**Transformation formulas:**
- \((r, \theta) \mapsto (x, y): x = r \cos \theta; \ y = r \sin \theta.\)
- \((x, y) \mapsto (r, \theta): r = \sqrt{x^2 + y^2}, \ y/x = \tan \theta\) (the value of \( 0 \leq \theta \leq 2\pi \) depends on the quadrant in which \((x, y)\) lies).

**Fact:** Suppose that \( D = \{(x, y) : a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\} = \{(r, \theta) : \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta)\}, \) and that \( f(x, y) \) is continuous on \( D \). Then we have
\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

**Note:** We have \( dy \, dx = dA = r \, dr \, d\theta \). The extra factor \( r \) arises as the absolute value of the Jacobian of the change of coordinates
\[
x(r, \theta) = r \cos \theta, \quad y(r, \theta) = r \sin \theta.
\]

The Jacobian is
\[
\begin{vmatrix}
\frac{\partial (x, y)}{\partial (r, \theta)} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r (\cos^2 \theta + \sin^2 \theta) = r.
\]
§15.6: Triple integrals.

**Definition:** Let $E$ be a simple solid in $\mathbb{R}^3$:

$$E = \{(x, y, z) : a \leq x \leq b, \ u_1(x) \leq y \leq u_2(x), \ v_1(x, y) \leq z \leq v_2(x, y)\},$$

and suppose that $f(x, y, z)$ is continuous on $E$. Then we have

$$\iiint_E f(x, y, z) \, dV = \int_a^b \int_{u_1(x)}^{u_2(x)} \int_{v_1(x,y)}^{v_2(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

**Note:** In rectangular coordinates $(x, y, z)$, there are six possible integration orders. Care must be taken when switching the order unless all limits of integration are constant.

**Possible strategy:** Once we pick an order, say $dV = dz \, dy \, dx$, then we may consider working in two steps.

1. We first try to identify $E$; at least we try to identify equations for the “top” of $E$: $z = v_2(x, y)$, and the “bottom” of $E$: $z = v_1(x, y)$. We find that

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[ \int_{v_1(x,y)}^{v_2(x,y)} f(x, y, z) \, dz \right] \, dA = \iint_D F(x, y) \, dA,$$

where $D$ is the projection (or “shadow”) of $E$ on the $xy$-plane.

2. Second, we evaluate the double integral (as in Sections §15.2 - 15.4)

$$\iint_D F(x, y) \, dA,$$

which may require a quick sketch of $D$ in the $xy$-plane.

**Fact:** Suppose that $E \subseteq \mathbb{R}^3$ is a solid which can be contained in a finite box. The volume of $E$ as a triple integral is

$$\text{Vol}(E) = \iiint_E dV.$$
§15.7: Triple integrals in cylindrical coordinates.

**Cylindrical coordinates:** $P = (r, \theta, z)$. (Let $Q = (x, y, 0)$ be the projection of $P$ in rectangular coordinates on the $xy$-plane.

- $r \geq 0$: the distance from the origin (pole), $O$, to the projection $Q$.
- $0 \leq \theta \leq 2\pi$: the angle, measured counter-clockwise, that the ray $OQ$ makes with the positive $x$-axis.
- $z \in \mathbb{R}$: the height of the point $P$ above the $xy$-plane.

**Transformation formulas:**

- $(r, \theta, z) \mapsto (x, y, z)$: $x = r \cos \theta; \ y = r \sin \theta; \ z = z$.
- $(x, y, z) \mapsto (r, \theta, z)$: $r = \sqrt{x^2 + y^2}; \ y/x = \tan \theta$ (the value of $0 \leq \theta \leq 2\pi$ depends on the quadrant in which $(x, y)$ lies); $z = z$.

**Fact:** Suppose that $E = \{(r, \theta, z) : \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta), \ v_1(r, \theta) \leq z \leq v_2(r, \theta)\}$, and that $f(x, y, z)$ is continuous on $E$. Then we have

$$\int \int \int_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{v_1(r, \theta)}^{v_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \, dr \, d\theta.$$

**Note:** We have $dV = r \, dz \, dr \, d\theta$. The extra factor $r$ arises as the absolute value of the Jacobian, just as it does for polar coordinates.

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = r$$

**Possible strategy:** As for triple integrals in rectangular $((x, y, z))$ coordinates, we work in two steps.

1. We try to identify equations for the “top” of $E$: $z = v_2(r, \theta)$, and the “bottom” of $E$: $z = v_1(r, \theta)$. We find that

$$\int \int \int_E f(x, y, z) \, dV = \int \int_D \left[ \int_{v_1(r, \theta)}^{v_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \right] \, dA = \int \int D F(r, \theta) \, dA.$$

2. We view the projection (or “shadow”) of $E$ in the $xy$-plane as a polar region:

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta)\},$$

and we compute the double integral in polar coordinates:

$$\int \int D F(r, \theta) \, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} F(r, \theta) r \, dr \, d\theta.$$

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§15.8: Triple integrals in spherical coordinates.

**Spherical coordinates:**  \( P = (\rho, \theta, \phi) \). (Let \( Q = (x, y, 0) \) be the projection of \( P \) in rectangular coordinates on the \( xy \)-plane.)

- \( \rho \geq 0 \): the distance from the origin, \( O \), to the point \( P \).
- \( 0 \leq \theta \leq 2\pi \): the angle, measured counter-clockwise, that the ray \( OQ \) makes with the positive \( x \)-axis.
- \( 0 \leq \phi \leq \pi \): the angle that the ray \( OP \) makes with the positive \( z \)-axis.

**Transformation formulas:**

- \((\rho, \theta, \phi) \mapsto (x, y, z): x = \rho \sin \phi \cos \theta; \ y = \rho \sin \phi \sin \theta; \ z = \rho \cos \phi.\)
- \((x, y, z) \mapsto (\rho, \theta, \phi): \rho = \sqrt{x^2 + y^2 + z^2}; \ \theta : \tan \theta = y/x \) (the value of \( \theta \) depends on the quadrant in which \((x, y)\) lies); \ \phi : \tan \phi = (x^2 + y^2)^{1/2}/z \) (the value of \( \phi \) depends the sign of \( z \)).

**Fact:** Suppose that

\[ E = \{(\rho, \theta, \phi) : \alpha \leq \theta \leq \beta, \ \phi_1(\theta) \leq \phi \leq \phi_2(\theta), \ \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)\}, \]

and that \( f(x, y, z) \) is continuous on \( E \). Then we have

\[
\iiint_E f(x, y, z) \, dV = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\theta, \phi)}^{\rho_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
\]

**Note:** We have \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \). The extra factor \( \rho^2 \sin \phi \) arises as the absolute value of the Jacobian of the change of coordinates

\[
x(\rho, \theta, \phi) = \rho \sin \phi \cos \theta, \ \ y(\rho, \theta, \phi) = \rho \sin \phi \sin \theta, \ \ z(\rho, \theta, \phi) = \rho \cos \phi,
\]

given by

\[
\frac{\partial (x, y, z)}{\partial (\rho, \theta, \phi)} = \begin{vmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{vmatrix} = -\rho^2 \sin \phi.
\]

**Possible strategy:** We sketch the solid \( E \), just as when we work in other coordinate systems. We use this sketch to determine all limits. The notion of a projection (or “shadow”) of \( E \) does not make sense in this system.