1. (53 points): Short Answer.

(a) (7 points): Suppose that \( \vec{u} \) is a unit vector, that \( \vec{v} \) is a vector with \( \|\vec{v}\| = 5 \), and that the angle \( \theta \) between \( \vec{u} \) and \( \vec{v} \) has \( \sin \theta = 3/5 \). Find the length of \( \vec{u} \times \vec{v} \).

**Solution:** We have

\[
\sin \theta = \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\||\|\vec{v}\|} \iff \|\vec{u} \times \vec{v}\| = \|\vec{u}\||\|\vec{v}\||\sin \theta = (1)(5)(3/5) = 3.
\]

(b) (7 points): Suppose that \( \vec{u} \) is a unit vector, and that \( \vec{v} \) is a vector with \( \vec{u} \cdot \vec{v} = 11 \). Suppose also that the angle \( \theta \) between \( \vec{u} \) and \( \vec{v} \) has \( \cos \theta = 4/7 \). Find the length of \( \vec{v} \).

**Solution:** We have

\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\||\|\vec{v}\|} \iff \|\vec{v}\| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\||\cos \theta} = \frac{11}{(1)(4/7)} = 77/4.
\]

(c) (7 points): Let \( \vec{v} = 2\hat{i} + \hat{j} - 3\hat{k} \) and \( \vec{u} = -\hat{i} - 2\hat{j} + \hat{k} \). Compute \( \text{proj}_{\vec{u}} \vec{v} \), the vector projection of \( \vec{v} \) onto \( \vec{u} \). Write your answer in the form \( a\vec{i} + b\vec{j} + c\vec{k} \). (To help remember the formula, recall that the projection onto \( \vec{u} \) points in the direction of \( \vec{u} \) or \( -\vec{u} \), so it is a scalar multiple of \( \vec{u} \).)

**Solution:** We have

\[
\text{proj}_{\vec{u}} \vec{v} = \left( \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \right) \vec{u} = \left( \frac{(2)(-1) + (1)(-2) + (-3)(1)}{(-1)^2 + (-2)^2 + 1^2} \right) (-\hat{i} - 2\hat{j} + \hat{k})
\]
\[
= \frac{7}{6}(-\hat{i} - 2\hat{j} + \hat{k}) = \frac{7}{6}\hat{i} + \frac{7}{3}\hat{j} - \frac{7}{6}\hat{k}.
\]

(d) (7 points): Let \( P = (2, 1, 4) \), \( Q = (1, 1, 1) \), and \( R = (1, 4, 3) \). Suppose that segments \( QP \) and \( QR \) are adjacent sides of parallelogram \( PQRS \). Find the coordinates of \( S \). (How does one add vectors geometrically?)

**Solution:** Let \( O = (0, 0, 0) \). Note that \( \overrightarrow{QR} = (0, 3, 2) \) and that \( \overrightarrow{QP} = (1, 0, 3) \).

- We have \( \overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{QR} = (2, 1, 4) + (0, 3, 2) = (2, 4, 6) \)
- We have \( \overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{QP} = (1, 4, 3) + (1, 0, 3) = (2, 4, 6) \)

Both give \( S = (2, 4, 6) \). This is the parallelogram law for vector addition.
Statement: The direction vectors are 

To obtain a point on 

We have 

Note that 

(20 points): 

10 

It follows that 

Q 

vector to 

Q 

and is perpendicular to the plane 

R 

Find an equation for the plane 

v 

Solution: 

(Identify two vectors, not parallel to each other, which are both parallel to the plane 

Q 

and is perpendicular to the plane 

R 

Solution: 

\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = \mathbf{u} \times \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \times \mathbf{w} \cdot \mathbf{v} = 0. \]

Solution: 

Statement II is false. For example, we have \( \vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j} \neq \vec{0}. \)

(8 points): Suppose that \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors in \( \mathbb{R}^3 \) with \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 3. \) What is the value of \( (\mathbf{w} \times \mathbf{v}) \cdot (5\mathbf{u})? \) Briefly explain.

Solution: We have

\[ (\mathbf{w} \times \mathbf{v}) \cdot (5\mathbf{u}) = (5\mathbf{u}) \cdot (\mathbf{w} \times \mathbf{v}) = (5\mathbf{u}) \cdot [-\mathbf{v} \times \mathbf{w}] = -5[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})] = -5(3) = -15. \]

(8 points): Suppose that \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal vectors in \( \mathbb{R}^3, \) and that \( \|\mathbf{u}\| = 7. \) What is the value of \( \mathbf{u} \cdot (2\mathbf{u} + 5\mathbf{v})? \) Briefly explain.

Solution: Note that \( \mathbf{u} \cdot \mathbf{v} = 0. \) Therefore, we have

\[ \mathbf{u} \cdot (2\mathbf{u} + 5\mathbf{v}) = 2\|\mathbf{u}\|^2 + 5\mathbf{u} \cdot \mathbf{v} = 2(7^2) = 98. \]

2. (12 points): Consider two lines in \( \mathbb{R}^3: \)

\[ L_1 : \begin{align*} x - 1 & = \frac{y - 6}{10} = \frac{z + 7}{-15}; \\
L_2 : & = \frac{x - 5}{8} = \frac{y + 2}{-12} = \frac{z - 3}{16}. \end{align*} \]

Determine whether they are parallel, perpendicular, or neither. Briefly explain.

Solution: The direction vectors are \( \vec{v}_1 = (10, -15, 20) \) and \( \vec{v}_2 = (8, -12, 16). \) Since \( \vec{v}_1 = \frac{5}{4}\vec{v}_2 \) we see that the lines are parallel.

3. (20 points): Find an equation for the plane \( Q \) that contains the line

\[ L : x = -1 + 3t, \ y = 1 + 2t, \ z = 2 + 4t, \ t \in \mathbb{R} \]

and is perpendicular to the plane \( R : \ 2x + y - 3z = -4. \) Write your answer in the form \( ax + by + cz = d. \)

(Identify two vectors, not parallel to each other, which are both parallel to the plane \( Q. \))

Solution: To obtain a point on \( Q, \) we set \( t = 0 \) in the equation for \( L: (-1, 1, 2). \) To obtain a normal vector to \( Q, \) we observe that the direction vector for \( L, \vec{v} = (3, 2, 4), \) and a normal vector to \( R, \vec{n} = (2, 1, -3), \) are both parallel to the plane \( Q. \) Hence, a normal vector to \( R \) is

\[ \vec{n} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\
2 & 1 & -3 \\
3 & 2 & 4 \end{vmatrix} = 10\vec{i} - 17\vec{j} + \vec{k}. \]

It follows that \( Q \) has equation \( 10(x + 1) - 17(y - 1) + (z - 2) = 0; \) equivalently, we have \( 10x - 17y + z = -25. \)
4. (15 points): Suppose you know that the planes

\[ Q_1 : 3x - 2y + z = 1, \quad Q_2 : 2x + y - 3z = 3 \]

intersect in a line \( L \), and that the point \( P = (1, 1, 0) \) is on the line \( L \). (You do not have to show that \( P \) is on \( L \).) Find parametric equations for \( L \).

**Solution:** We only need a direction vector, which we obtain as the cross product of the normal vectors of the planes:

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 5\vec{i} + 11\vec{j} + 7\vec{k}.
\]

Hence, \( L \) has parametric equations

\[
x = 1 + 5t, \quad y = 1 + 11t, \quad z = 7t, \quad t \in \mathbb{R}.
\]