Math 241, Exam 1 Information.
9/19/11, LC 121, 11:15 - 12:05.

Exam 1 will be based on:

- Sections 12.1 - 12.5.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/~boylan/SCCourses/241Fa11/241.html)
  At minimum, you need to understand how to do the homework problems.

Topic List (not necessarily comprehensive):

You will need to know: theorems, results, and definitions from class.

§12.1: 3-dimensional space.
Distance formula: \( d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \).

Sphere with center \((x_0, y_0, z_0)\) and radius \(r\):
\[
\{ (x, y, z) : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \}.
\]

§12.2: Vectors.

Vectors have length (magnitude) and direction.

Operations:

1. Addition.
   (a) Algebraically: \( \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \).
   (b) Geometrically: Parallelogram Law.
   (c) Properties: closure, zero element, associativity, inverses \((-\vec{u} = (-1)\vec{u})\), commutativity.

2. Scalar multiplication.
   (a) Algebraically: \( k\langle u_1, u_2, u_3 \rangle = \langle ku_1, ku_2, ku_3 \rangle \).
   (b) Geometrically: \( k\vec{u} \) scales \( \vec{u} \) by a factor of \(|k|\); if \( k < 0 \), then \( k\vec{u} \) points in the direction opposite \( \vec{u} \).
   (c) Properties: closure, associativity, distributivity.
Length. \[ \| \langle v_1, v_2, v_3 \rangle \| = \sqrt{v_1^2 + v_2^2 + v_3^2} \]

Notes.
1. For all \( \vec{v} \in \mathbb{R}^3 \), we have \( \| \vec{v} \| \geq 0 \) and \( \| \vec{v} \| = 0 \iff \vec{v} = \vec{0} \).
2. For \( k \in \mathbb{R} \), we have \( \| k\vec{v} \| = |k|\|\vec{v}\| \).

Unit vectors. A vector \( \vec{v} \) is a unit vector if and only if \( \| \vec{v} \| = 1 \).
1. Standard basis unit vectors: \( \vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle \).
2. The normalization of \( \vec{v} \in \mathbb{R}^3 \) is the unit vector \( \vec{v} / \| \vec{v} \| \).

§12.3: The dot product.
Definition: \( \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3 \). The input consists of two vectors; the output is a scalar.

Properties.
1. Commutative, distributive.
2. \( \vec{u} \cdot \vec{u} = \| \vec{u} \|^2 \).
3. Let \( 0 \leq \theta \leq \pi \) be the angle between vectors \( \vec{u} \) and \( \vec{v} \). Then we have \( \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \| \| \vec{v} \|} \).

Orthogonality. Two vectors \( \vec{u} \) and \( \vec{v} \) are orthogonal if and only if \( \vec{u} \cdot \vec{v} = 0 \); geometrically, this holds if and only if the vectors are perpendicular: the angle between them is \( \pi / 2 \).
1. The orthogonal projection of \( \vec{v} \) on \( \vec{b} \) is \( \text{proj}_b \vec{v} = \left( \frac{\vec{v} \cdot \vec{b}}{\| \vec{b} \|^2} \right) \vec{b} \).
2. The component of \( \vec{v} \) on \( \vec{b} \) is \( \text{comp}_b \vec{v} = \| \text{proj}_b \vec{v} \| = \frac{\vec{v} \cdot \vec{b}}{\| \vec{b} \|} \).

Triangle inequality: \( \| \vec{u} + \vec{v} \| \leq \| \vec{u} \| + \| \vec{v} \| \).
§12.4: The cross product.

**Definition:** Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then we have

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3)\vec{i} - (u_1v_3 - v_1u_3)\vec{j} + (u_1v_2 - v_1u_2)\vec{k}$$

Two vectors are input; the output is a vector.

**Facts.**

1. Distributive.
2. Not associative in general.
3. $\vec{u} \times \vec{u} = \vec{0}$.
4. Anti-commutative: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
5. $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$. The other products of these vectors are obtainable from anti-commutativity.

**Geometry.**

1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Of two possible directions, the correct one is obtained via the right-hand rule.
2. Let $0 \leq \theta \leq \pi$ be the angle between $\vec{u}$ and $\vec{v}$. Then we have $\sin \theta = \frac{||\vec{u} \times \vec{v}||}{||\vec{u}|| ||\vec{v}||}$.
3. The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is $||\vec{u} \times \vec{v}||$.
4. The vectors $\vec{u}$ and $\vec{v}$ are parallel ($\vec{u} \parallel \vec{v}$) if and only if $\vec{u} \times \vec{v} = \vec{0}$.

**Triple product:** Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$. Then we have

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$ 

Moreover, this value gives the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$. 

§12.5: Equations of lines and planes.

I. Lines. A line $L$ is determined by a direction vector $\mathbf{v} = \langle a, b, c \rangle$ and a point $P_0 = (x_0, y_0, z_0)$ on the line. The equation of a line can take several forms

(a) **Parametric form.** $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad t \in \mathbb{R}$.

(b) **Vector form.** $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Here, we have $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

(c) **Symmetric form.** Suppose that none of $a, b, c$ is zero. Then we have

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$ Suitable adjustments are required when one or more of the components of $\mathbf{v}$ is zero.

**Line segments.** A line segment is a finite portion of a line. Let $P_1 = (x_1, y_1, z_1)$, and $P_2 = (x_2, y_2, z_2)$; let $\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$, and let $\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$. Then the line segment from $P_1$ to $P_2$ has vector equation

$$\mathbf{r} = \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)t, \quad 0 \leq t \leq 1.$$

**Skew lines.** Two lines in $\mathbb{R}^3$ are skew if and only if they are non-parallel lines lying in parallel planes.

**Facts.** Let $L_1$ and $L_2$ be lines with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(a) The lines $L_1$ and $L_2$ are parallel if and only if their direction vectors are; this happens if and only if there exists $k \in \mathbb{R}$ with $k\mathbf{v}_1 = \mathbf{v}_2$.

(b) The lines $L_1$ and $L_2$ are perpendicular if and only if their direction vectors are orthogonal: $\mathbf{u} \cdot \mathbf{v} = 0$.

(c) Two lines are skew if they are non-intersecting and non-parallel. To determine whether two non-parallel lines intersect requires one to solve a system of three equations in two variables. If the system has a solution, the lines intersect at the point given by the solution; if the system has no solution, then the lines are skew.

II. Planes. A plane $Q$ is determined by a normal vector $\mathbf{n} = \langle a, b, c \rangle$ which measures the “tilt” of $Q$ in $\mathbb{R}^3$ and a point $P_0 = (x_0, y_0, z_0)$ on $Q$. The normal vector is perpendicular to $Q$. The equation of the plane $Q$ is

(a) $Q : a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

(b) $Q : ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.

**Observations.**

(a) Planes $Q_1$ and $Q_2$ are parallel if and only if their normal vectors are: $\mathbf{n}_1 \parallel \mathbf{n}_2$.

(b) Planes $Q_1$ and $Q_2$ are perpendicular if and only if their normal vectors are: $\mathbf{n}_1 \perp \mathbf{n}_2$.

(c) Line $L$ is parallel to plane $Q$ if and only if the direction vector of $L$ is perpendicular to the normal of $Q$: $\mathbf{n} \perp \mathbf{v}$.

(d) Line $L$ is perpendicular to plane $Q$ if and only if the direction vector of $L$ is parallel to the normal of $Q$: $\mathbf{n} \parallel \mathbf{v}$. 
Distance. Let $P_0 = (x_0, y_0, z_0)$ and let $Q : ax + by + cz + d = 0$. Then the distance from $P_0$ to $Q$ is

$$d(P_0, Q) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

One can use this formula to find the distance between the following objects (assumed to be non-intersecting so the distance makes sense):

- a point and a plane;
- a point and a line;
- a line and a plane;
- two planes;