§14.8, #33 (15 points): Use Lagrange Multipliers to find the maximum volume of a rectangular box that is inscribed in a sphere of radius $r$.

**Solution:** The problem is to maximize $V = (2x)(2y)(2z) = 8xyz$ subject to the constraint $g = x^2 + y^2 + z^2 = r^2$. We compute

$$\nabla V = \langle 8yz, 8xz, 8xy \rangle, \quad \nabla g = \langle 2x, 2y, 2z \rangle.$$

With $\lambda \neq 0$ and $x, y, z > 0$ (as spatial dimensions), we solve the system

$$4yz = \lambda x, \quad 4xz = \lambda y, \quad 4xy = \lambda z, \quad x^2 + y^2 + z^2 = r^2.$$

Since $x, y, z \neq 0$, we solve for $\lambda$ in the first three equations to obtain

$$\frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z} = \lambda.$$

We take equations (1) and (2) together to get $4y^2z = 4x^2z$. Since $z \neq 0$, we find that $y^2 = x^2$. Similarly, using equations (1) and (3) gives $x^2 = z^2$; we conclude that $x^2 = y^2 = z^2$. We substitute in the fourth equation in the original system to get $3x^2 = r^2$. Taking the positive square root gives $x = y = z = r/\sqrt{3}$; hence, the volume is $V = 8(r/\sqrt{3})^3 = 8r^3/3\sqrt{3}$. 