Math 241, Quiz 12. 11/19/12. Name: ______________________

- Read problems carefully. Show all work. No notes, calculator, or text.
- There are 15 points total.

1. §15.8, #25 (7 points): Use spherical coordinates to set up \( \iiint_E x^2 \, dV \), where \( E \) is bounded by \( xz \)-plane and the hemispheres \( y = \sqrt{9 - x^2 - z^2} \) and \( y = \sqrt{16 - x^2 - z^2} \).
   Do not evaluate.

   **Solution:** We have
   \[
   \int_0^\pi \int_0^\pi \int_3^4 (\rho^2 \sin \phi \cos \theta)^2 (\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta).
   \]
   \[
   = \int_0^\pi \int_0^\pi \int_3^4 \rho^4 \sin^3 \phi \cos^2 \theta \, d\rho \, d\phi \, d\theta.
   \]

2. §15.8, #40 (8 points): Evaluate the integral
   \[
   \int_{-a}^a \int_{\sqrt{a^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_{\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2 z + y^2 z + z^3) \, dz \, dx \, dy, \; a \geq 0.
   \]
   by changing to spherical coordinates.

   **Solution:** First, we note that \( x^2 z + y^2 z + z^3 = (x^2 + y^2)z + z^3 = (x^2 + y^2 + z^2)z = \rho^2 z = \rho^2 (\rho \cos \phi) = \rho^3 \cos \phi \). We have
   \[
   \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \cos \phi (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^5 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta
   \]
   \[
   = \left( \int_0^{2\pi} d\theta \right) \left( \frac{1}{2} \int_0^\pi 2 \sin \phi \cos \phi \, d\phi \right) \left( \int_0^a \rho^5 \, d\rho \right) = (2\pi) \left( \frac{1}{2} \int_0^\pi \sin 2\phi \, d\phi \right) \left( \frac{a^6}{6} \right)
   \]
   \[
   = \frac{\pi a^6}{6} \left( -\frac{1}{2} \cos 2\phi \right)_{\pi}^{0} = -\frac{\pi a^6}{12} (\cos 2\pi - \cos 0) = -\frac{\pi a^6}{12} \cdot 0 = 0.
   \]