The Dual-Tree Complex Wavelet Transform

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Outline

Signal Processing
   Signals and Filters
   Subband Filtering Schemes with Perfect Reconstruction

Wavelets and Subband Filtering Schemes
   Wavelets: Multiresolution Analysis
   Connection with Subband Filtering Schemes

Problems with Real Wavelets
   Shift Variance
   Lack of directionality

Solution
   Complex Wavelets
   The Dual-Tree Framework
Any square summable sequence \((c_n)_{n \in \mathbb{Z}}\) can be interpreted as the sequence of sampled values \(f(n)\) of a band-limited function \(f\) with \(\text{supp} \hat{f} \subset [-\pi, \pi]\).

\[
f(x) = \sum_{n \in \mathbb{Z}} c_n \frac{\sin \pi(x - n)}{\pi(x - n)}, \quad \hat{f}(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{-in\xi}.
\]
Background material. Notation and Examples

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\]

A filtering operation corresponds to the multiplication of \(\hat{f}\) with a \(2\pi\)-periodic function, \(\hat{\alpha}(\xi) = \sum_{n \in \mathbb{Z}} a_n e^{-in\xi}\). The result is another band-limited function:

\[
(\alpha \ast f)(x) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_{n-m}c_m \frac{\sin \pi(x - n)}{\pi(x - n)},
\]

\[
\mathcal{F}(\alpha \ast f)(\xi) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_{n-m}c_m e^{-in\xi}.
\]
Example: 3-by-3 unsharp contrast enhancement filter

\[ f(x) \xrightarrow{\text{signal}} \hat{\alpha}(\xi) \xrightarrow{\text{transfer function / frequency response}} (\alpha \ast f)(x) \]

\((c_n)_{n \in \mathbb{Z}} \xrightarrow{\text{filter / impulse response}} (a_n)_{n \in \mathbb{Z}} \xrightarrow{\text{signal}} \left( \sum_{m \in \mathbb{Z}} a_m - n c_m \right)_{n \in \mathbb{Z}} \)
Subband Filtering Schemes

In signal processing an incoming signal is often decomposed into different frequency bands after which they can then be coded and transmitted separately and efficiently. This decomposition of a signal is usually done using a collection of filters called a filter bank.
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Reconstruction

\[
\begin{array}{c}
\tilde{c}_n \\
\hline
\uparrow 2 \\
\rightarrow \quad \rightarrow \\
\downarrow 2 \\
\rightarrow \quad \rightarrow \\
\tilde{a}_n^1 \\
\downarrow 2 \\
\rightarrow \quad \rightarrow \\
\tilde{a}_n^0 \\
\uparrow 2 \\
\rightarrow \quad \rightarrow \\
a_n^0 \\
\uparrow 2 \\
\rightarrow \quad \rightarrow \\
c_n
\end{array}
\]
Reconstruction

\[ c_n \rightarrow a_n^0 \rightarrow \downarrow 2 \rightarrow \uparrow 2 \rightarrow a_n^0 \rightarrow \tilde{c}_n \]

\[ \sum c_n e^{-in\xi} \]

\[ \sum \tilde{c}_n e^{-in\xi} \]
Reconstruction

\[ c(z) = \sum c_n z^{-n} \]

\[ \tilde{c}(z) = \sum \tilde{c}_n z^{-n} \]
Reconstruction

\[ c(z) = \sum c_n z^{-n} \]

\[ \frac{1}{2} \left( a^0(z) c(z) + a^0(-z) c(-z) \right) \]

\[ \frac{1}{2} \left( a^1(z) c(z) + a^1(-z) c(-z) \right) \]
Reconstruction

\[ c(z) = \sum c_n z^{-n} \]

\[ \tilde{c}(z) = \sum \tilde{c}_n z^{-n} \]

\[ \tilde{c}(z) = \frac{1}{2} \left( \tilde{a}^0(z) a^0(z) + \tilde{a}^1(z) a^1(z) \right) c(z) + \frac{1}{2} \left( \tilde{a}^0(z) a^0(-z) + \tilde{a}^1(z) a^1(-z) \right) c(-z) \]

aliasing effects
Perfect Reconstruction

\[ c(z) = \sum c_n z^{-n} \]

\[ a^0(z) \Downarrow 2 \rightarrow a^0_n \rightarrow \Downarrow 2 \rightarrow a^0(z) \]

\[ a^1(z) \Downarrow 2 \rightarrow a^1_n \rightarrow \Downarrow 2 \rightarrow a^1(z) \]

\[ \tilde{a}^0(z) \Downarrow 2 \rightarrow \tilde{a}^0_n \rightarrow \Downarrow 2 \rightarrow \tilde{a}^0(z) \]

\[ \tilde{a}^1(z) \Downarrow 2 \rightarrow \tilde{a}^1_n \rightarrow \Downarrow 2 \rightarrow \tilde{a}^1(z) \]

\[ \frac{1}{2} (a^0(z)c(z) + a^0(-z)c(-z)) \]

\[ \frac{1}{2} (a^1(z)c(z) + a^1(-z)c(-z)) \]

\[ \tilde{c}_n \]

\[ \tilde{c}(z) = \sum \tilde{c}_n z^{-n} \]

\[ \tilde{a}^0(z)a^0(z) + \tilde{a}^1(z)a^1(z) = 2, \quad \tilde{a}^0(z)a^0(-z) + \tilde{a}^1(z)a^1(-z) = 0 \]
1,024 × 1,024 = 1,048,576 pixels
Wavelets

1,024 \times 1,024 = 1,048,576 \text{ pixels}

1 \text{ wavelet coefficient}
Wavelets

$1,024 \times 1,024 = 1,048,576$ pixels

$1 + 4 = 5$ wavelet coefficients
1,024 \times 1,024 = 1,048,576 \text{ pixels}

1 + 4 + 16 = 21 \text{ wavelet coefficients}
Wavelets

1,024 × 1,024 = 1,048,576 pixels

1 + 4 + 16 + 64 = 85 wavelet coefficients
Wavelets

1,024 × 1,024 = 1,048,576 pixels

1 + 4 + 16 + 64 + 256 = 341 wavelet coefficients
Wavelets

\[ 1,024 \times 1,024 = 1,048,576 \text{ pixels} \]

21,845 wavelet coefficients
Connection with Subband Filtering Schemes

Multiresolution analysis leads to a hierarchical scheme for the computation of the wavelet coefficients of a function:
Problems with Real Wavelets

Shift Variance
A small shift of the signal causes major variations in the distribution of energy between wavelet coefficients at different scales.
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Poor Directional Selectivity
The standard tensor-product construction of multi-variate wavelets produces a checkerboard pattern that is simultaneously oriented along several directions. This lack of directional selectivity complicates processing of geometric image features like ridges and edges.

N. Kingsbury: “Complex Wavelets for Shift Invariant Analysis and Filtering of Signals”
Some Complex Wavelets do not have those problems!

N. Kingsbury: “Complex Wavelets for Shift Invariant Analysis and Filtering of Signals”

The key: “Hilbert Transform pairs”

Complex-valued scaling function \( \phi: \mathbb{R} \rightarrow \mathbb{C} \) and complex-valued wavelet \( \psi: \mathbb{R} \rightarrow \mathbb{C} \) satisfying

\[
\psi(t) = u(t) + i\mathcal{H}u(t).
\]

You get extra points if \( u: \mathbb{R} \rightarrow \mathbb{R} \) is even, and \( \mathcal{H}u \) is odd.
Watch out! Not so easy to code

For a complex-valued function \( \psi(t) = u(t) + i\mathcal{H}u(t) \),

\[
\hat{\psi}(\xi) = \hat{u}(\xi) + i\mathcal{F}(\mathcal{H}u)(\xi)
= \hat{u}(\xi) - \text{sign}(\xi)\hat{u}(\xi)
= \begin{cases} 
0 & \text{if } \xi > 0, \\
\hat{u}(0) & \text{if } \xi = 0, \\
2\hat{u}(\xi) & \text{if } \xi < 0.
\end{cases}
\]
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\end{cases}$$

Neither $a^0(z)$ nor $\tilde{a}^0(x)$ is a reasonable low-pass filter.
The Dual-Tree CWT

The idea:
Require $u(t)$ to be a real-valued wavelet such that $\mathcal{H}u(t)$ is also a wavelet, and perform two different subband filtering schemes for real and imaginary parts independently.
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- The filters are real; no complex arithmetic is required for the implementation.
- The dual-tree CWT is not critically sampled: it is two times expansive in 1-D.
- The inverse is simple: real and imaginary parts are inverted to obtain two real signals. These two signals are then averaged.