Exam #1

SOLUTIONS

1. { 15 points }  Find the total area of the region between the curves \( y = x^2 \) and \( y = \sin \left( \frac{\pi}{2} x \right) \) for \(-1 \leq x \leq 1\).

\[
A = \int_{-1}^{0} x^2 - \sin \left( \frac{\pi}{2} x \right) \, dx + \int_{0}^{1} \sin \left( \frac{\pi}{2} x \right) - x^2 \, dx
\]

\[
= \frac{x^3}{3} \bigg|_{-1}^{0} + \frac{2}{\pi} \cos \left( \frac{\pi}{2} x \right) \bigg|_{0}^{1} - \frac{2}{\pi} \cos \left( \frac{\pi}{2} x \right) \bigg|_{0}^{1} - \frac{x^3}{3} \bigg|_{0}^{1}
\]

\[
= \frac{1}{3} (0 - (-1)) + \frac{2}{\pi} (1 - 0) - \frac{2}{\pi} (0 - 1) - \frac{1}{3} (1 - 0) = \frac{4}{\pi}
\]

2. { 15 points }  Find the total area of the region enclosed by the curves \( y = 7 - x \) and \( y = \sqrt{25 - 4x} \).

To find the intersection points of the curves we solve

\( 7 - x = \sqrt{25 - 4x} \).

Squaring both sides of the equation we receive

\( 49 - 14x + x^2 = 25 - 4x \quad \Leftrightarrow \quad x^2 - 10x + 24 = 0 \)

which have solutions \( x = 4 \) and \( x = 6 \).

In addition, we see that \( \sqrt{25 - 4x} \geq 7 - x \). Thus,

\[
A = \int_{4}^{6} \sqrt{25 - 4x} - (7 - x) \, dx = \frac{1}{4} \int_{4}^{6} \sqrt{25 - 4x} \, d(25 - 4x) + \int_{4}^{6} x - 7 \, dx
\]

\[
= -\frac{1}{4} \left( \frac{2}{3} \right) (25 - 4x)^{\frac{3}{2}} \bigg|_{4}^{6} + \left( \frac{x^2}{2} - 7x \right) \bigg|_{4}^{6}
\]

\[
= -\frac{1}{6} \left( (25 - 24)^{\frac{3}{2}} - (25 - 16)^{\frac{3}{2}} \right) + \frac{1}{2} (6^2 - 4^2) - 7(6 - 4)
\]

\[
= -\frac{1}{6} \left( 1^{\frac{3}{2}} - 9^{\frac{3}{2}} \right) + \frac{1}{2} (36 - 16) - 14 = -\frac{1}{6} (1 - 27) + 10 - 14
\]

\[
= \frac{26}{6} - 4 = \frac{13}{3} - \frac{12}{3} = \frac{1}{3}
\]
3. { 15 points } Let \( R \) be the region between the curves \( y = \sqrt{x} \) and \( y = \frac{2}{\sqrt{x}} \) for \( 1 \leq x \leq 4 \). Find and evaluate a definite integral that represents the volume of the solid generated by revolving \( R \) about the \( x \)-axis.

Since \( \sqrt{x} \leq \frac{2}{\sqrt{x}} \) on the interval \([1, 2]\) and \( \sqrt{x} \geq \frac{2}{\sqrt{x}} \) on \([2, 4]\), we have

\[
V = \pi \int_{1}^{2} \left( \frac{2}{\sqrt{x}} \right)^{2} - (\sqrt{x})^{2} \, dx + \pi \int_{2}^{4} (\sqrt{x})^{2} - \left( \frac{2}{\sqrt{x}} \right)^{2} \, dx
\]

\[
= \pi \int_{1}^{2} \left( \frac{4}{x} - x \right) \, dx + \pi \int_{2}^{4} \left( x - \frac{4}{x} \right) \, dx
\]

\[
= \pi \left( 4 \ln 2 - \frac{1}{2} (4 - 1) + \frac{1}{2} (16 - 4) - 4 \ln 4 + 4 \ln 2 \right)
\]

\[
= \left( 4 \ln 2 - \frac{3}{2} + 6 - 4 \ln 2 \right) = \frac{9\pi}{2}
\]

4. { 15 points } Consider the region \( R \) enclosed between \( y = \sqrt{9 - x^2} \) and \( y = 2 - x \) for \( 0 \leq x \leq 2 \). Find the volume of the solid generated by revolving \( R \) about the \( y \)-axis by integrating with respect to \( x \).

We use cylindrical shells with heights \( \sqrt{9 - x^2} - (2 - x) \geq 0 \) for \( 0 \geq x \geq 2 \)

\[
V = 2\pi \int_{0}^{2} x \left( \sqrt{9 - x^2} - (2 - x) \right) \, dx
\]

\[
= 2\pi \int_{0}^{2} x \sqrt{9 - x^2} \, dx - 2\pi \int_{0}^{2} x (2 - x) \, dx
\]

\[
= 2\pi \left( -\frac{1}{2} \right) \int_{0}^{2} \sqrt{9 - x^2} (-2x) \, dx + 2\pi \int_{0}^{2} x^2 - 2x \, dx
\]

Change \( u = 9 - x^2 \) in the first integral. Then \( du = -2x \, dx \) and the limits change as follows: \( x = 0 \rightarrow u = 9 \) and \( x = 2 \rightarrow u = 9 - 2^2 = 5 \)

\[
V = -\pi \int_{9}^{5} \sqrt{u} \, du + 2\pi \left( \frac{x^3}{3} - x^2 \right) \bigg|_{0}^{5} = -\pi \frac{2}{3} u^{\frac{3}{2}} \bigg|_{9}^{5} + 2\pi \left( \frac{1}{3} (2^3 - 0^3) - (2^2 - 0^2) \right)
\]

\[
= -\frac{2\pi}{3} \left( 5\sqrt{5} - 9\sqrt{9} \right) + 2\pi \left( \frac{8}{3} - 4 \right) = -\frac{10\sqrt{5}\pi}{3} + 18\pi - \frac{8\pi}{3} = \frac{46\pi}{3} - \frac{10\sqrt{5}\pi}{3}
\]
5. { 15 points } Find the exact arc length of the parametric curve without eliminating the parameter

\[ x = \cos t + t \sin t , \quad y = \sin t - t \cos t \quad (0 \leq t \leq \pi) \]

\[ \begin{align*}
  x' &= -\sin t + \sin t + t \cos t = t \cos t \\
  y' &= \cos t - \cos t + t \sin t = t \sin t
\end{align*} \]

\[ L = \int_0^\pi \sqrt{(x')^2 + (y')^2} \, dt = \int_0^\pi \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} \, dt \]

\[ = \int_0^\pi t \sqrt{t^2} \, dt = \int_0^\pi t \, |t| \, dt = \int_0^\pi t \, t \, dt = \left. \frac{t^2}{2} \right|_0^\pi = \frac{\pi^2}{2} \]

6. { 15 points } Let \( C \) be the curve \( x - y^2 = 0 \) between \( x = 0 \) and \( x = 2 \). Find the area of the surface generated by revolving \( C \) about the \( x \)-axis by integrating with respect to \( x \).

\[ y^2 = x \quad \Rightarrow \quad y(x) = x^{\frac{1}{2}} \quad \Rightarrow \quad y'(x) = \frac{1}{2} x^{-\frac{1}{2}} \]

Thus,

\[ A = 2\pi \int_0^2 y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^2 \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx \]

\[ = 2\pi \int_0^2 \sqrt{x + \frac{1}{4}} \, dx = 2\pi \frac{2}{3} \left( x + \frac{1}{4} \right)^{\frac{3}{2}} \bigg|_0^2 \]

\[ = \frac{4\pi}{3} \left( \left( \frac{9}{4} \right)^{\frac{3}{2}} - \left( \frac{1}{4} \right)^{\frac{3}{2}} \right) = \frac{4\pi}{3} \left( \frac{27}{8} - \frac{1}{8} \right) = \frac{13\pi}{3} \]
7. \{ 15 \text{ points} \} \text{ Find the average value of } f(x) = \frac{1}{e^{-x}+e^{x}} \text{ over the interval } \left[ -\frac{\ln 3}{2}, 0 \right].

\[ f_{\text{ave}} = 0 - \left( -\frac{\ln 3}{2} \right) \int_{-\frac{\ln 3}{2}}^{0} \frac{1}{e^{-x}+e^{x}} \, dx = \frac{2}{\ln 3} \int_{\ln(\frac{1}{\sqrt{3}})}^{0} \frac{e^{x}}{e^{x} + e^{-x}} \, dx \]

\[ = \frac{2}{\ln 3} \int_{\ln(\frac{1}{\sqrt{3}})}^{0} \frac{e^{x}}{1 + e^{2x}} \, dx \]

\text{change of variables: } u = e^x \quad \Rightarrow \quad du = e^x \, dx \quad \text{with limits}\]

\[ x = \ln\left( \frac{1}{\sqrt{3}} \right) \quad \rightarrow \quad u = e^{\ln(\frac{1}{\sqrt{3}})} = \frac{1}{\sqrt{3}} \quad \text{and} \quad x = 0 \rightarrow u = e^{0} = 1 \]

\[ f_{\text{ave}} = \frac{2}{\ln 3} \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1 + u^2} \, du = \frac{2}{\ln 3} \tan^{-1} u \bigg|_{\frac{1}{\sqrt{3}}}^{1} = \frac{2}{\ln 3} \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{\pi}{6\ln 3} \]

8. \{ 15 \text{ points} \} \text{ A boat is anchored so that the anchor is } 100 \text{ ft} \text{ below the surface of the water. In the water, the anchor weights } 1200 \text{ lb} \text{ and the chain weights } 40 \text{ lb/ft}. \text{ How much work is required to raise the anchor to the surface?}

\text{When the anchor is } x \text{ ft} \text{ below the surface the total weight is } 1200 + 40x \text{ lb} \text{ and therefore the force is } F(x) = (1200 + 40x) \, g \text{, where } g \text{ is the gravity constant. Thus,}

\[ W = \int_{0}^{100} F(x) \, dx = \int_{0}^{100} (1200 + 40x)g \, dx = (20x^2 + 1200x) \bigg|_{0}^{100} \]

\[ = (200,000 + 120,000) \, g = 320,000 \, g \]