1. { 1 point for each correct answer } For each of the following statements determine whether it is true or false. It is not necessary to justify your answer.

(A) The square of any integer has the form $4k$ or $4k+1$ for some integer $k$. **true**
For even integers $n = 2p$ we have $n^2 = 4p^2 = 4k$. For odd integers $n = 2p+1$ we have $n^2 = 4p^2 + 4p + 1 = 4(p^2 + p) + 1 = 4k + 1$.

(B) For all real numbers $x$, $[x^2] = [x]^2$. **false**
Counterexample: for $x = 1.5$ we have $[x^2] = [2.25] = 2 \neq 1 = [1.5]^2 = 1^2$.

(C) For all real numbers $x$, $[x+1] = [x] + 1$. **true**
Let $n = [x]$. Then by the definition $n - 1 < x \leq n$. Hence $n < x + 1 \leq n + 1$ and therefore $[x + 1] = n + 1 = [x] + 1$.

(D) There is no least positive rational number. **true**
Suppose not. Let $x = \frac{a}{b} > 0$ be the least positive rational number, where $a$ and $b \neq 0$ are integers. Then $\frac{x}{2} = \frac{a}{2b} > 0$ is rational and $0 < \frac{x}{2} < x$, which is a contradiction.

(E) For all integers $a$ and $n$, if $a \mid n^2$ then $a \mid n$. **false**
Counterexample: for $a = 4$ and $n = 2$ we have $a = 4 \mid 4 = n^2$ but $a = 4 \not\mid 2 = n$.

2. { 5 points } Prove the statement. Recall that the symbol $\not\mid$ means ”does not divide”.

For all integers $a$, $b$, and $c$, if $a \not\mid bc$ then $a \not\mid b$.

**Suppose not.**

Then there exist integers $a$, $b$, and $c$ such that $a \not\mid bc$ and $a \mid b$.

From the definition of $a \mid b$ there exists an integer $k$ such that $b = ka$. Then $bc = kac = (kc)a$.

But $kc$ is an integer. Therefore $a \mid bc$, which is a contradiction.

The statement can be proved by contraposition. The contrapositive statement is:

For all integers $a$, $b$, and $c$, if $a \mid b$ then $a \mid bc$.

To prove of the last statement we assume that $a$, $b$, and $c$ are arbitrary integers and $a \mid b$.

Then there exists an integer $k$ such that $b = ka$. Thus $bc = kac = (kc)a$.

Since $kc$ is an integer, we have $a \mid bc$.

**Bonus Problem.** { 5 points } Prove the statement.

$4\sqrt{2} - 9$ is irrational.

**Suppose not, i.e. the number is rational.**

Then there exist integers $a$ and $b \neq 0$ such that $4\sqrt{2} - 9 = \frac{a}{b}$.

Hence $4\sqrt{2} = \frac{a}{b} + 9 = \frac{a+9b}{b}$ and therefore $\sqrt{2} = \frac{a+9b}{4b}$.

But $a + 9b$ and $4b \neq 0$ are integers.

Thus, $\sqrt{2}$ is a rational number, which is a contradiction.