SOLUTIONS for Exam # 2

1. { 10 points } **Find** $f'(x)$ **and** $f''(x)$.

$$f(x) = x^7 - 4\sqrt{x} + \frac{1}{2}\, x^{-2}$$

$$f'(x) = 7x^6 - 2x^{-\frac{1}{2}} - x^{-3}$$
$$f''(x) = 42x^5 + x^{-\frac{3}{2}} + 3x^{-4}$$

2. { 10 points } **Find** $f'(x)$.

$$f(x) = \frac{e^{-2x}}{x^2 + \cos^5 x}$$

$$f'(x) = \frac{-2e^{-2x}(x^2 + \cos^5 x) - e^{-2x}(2x + 5\cos^4 x(-\sin x))}{(x^2 + \cos^5 x)^2}$$
$$= \frac{e^{-2x}\left(-2x^2 - 2\cos^5 x - 2x + 5\cos^4 x \ \sin x\right)}{(x^2 + \cos^5 x)^2}$$

3. { 10 points } **Find** $\frac{dy}{dx}$.

$$y = xe^{\sqrt{x}} + \ln x$$

$$\frac{dy}{dx} = e^{\sqrt{x}} + xe^{\sqrt{x}}\left(\frac{1}{2}\, x^{-\frac{1}{2}}\right) + \frac{1}{x} = e^{\sqrt{x}} + \frac{\sqrt{x}}{2}e^{\sqrt{x}} + \frac{1}{x}$$

4. { 10 points } **Find** $\frac{dy}{dx}$.

$$y = \sqrt{\ \cos^{-1}(x^3)}$$

$$\frac{dy}{dx} = \frac{1}{2}\ \left(\cos^{-1}(x^3)\right)^{-\frac{1}{2}} \left(-\frac{1}{\sqrt{1-(x^3)^2}}\right)(3x^2) = \frac{-\,3x^2}{2\sqrt{\cos^{-1}(x^3)}\ \sqrt{1-x^6}}$$

5. { 10 points } **Find the values of x at which the curve $y = f(x)$ has a horizontal tangent line, if** $f(x) = (2x+5)^2(x-1)^6$.

Horizontal line means slope $m = 0$. Hence, we have to find all points x for which $f'(x) = 0$.

$$f'(x) = 2(2x+5)(2)\ (x-1)^6 \ + \ (2x+5)^2\ 6(x-1)^5$$

$$= (2x+5)(x-1)^5(4(x-1) + 6(2x+5)) = (2x+5)(x-1)^5(16x+26).$$

Thus, $x = -\frac{2}{5}$, $x = 1$, and $x = -\frac{26}{16} = -\frac{13}{8}$ are the points for which the tangent is horizontal.

6. { 15 points } **Complete each part for the function $f(x) = x^2 - 4x$.**

(a) **Find the slope of the tangent line to the graph of f at a general x-value.**

$$m = f'(x) = 2x - 4$$

(b) **Find the tangent line to the graph of f at $x = 1$.**

The equation of the tangent line is $y - y_0 = m(x - x_0)$

where $x_0 = 1$, $y_0 = f(x_0) = 1 - 4 = -3$, and $m = f'(x_0) = 2 - 4 = -2$.

Substituting these values we receive $y-(-3) = -2(x-1)$ and therefore $y = -2x - 1$ **is the equation of the tangent line.**

7. { 15 points } **Find $\frac{dy}{dx}$ by implicit differentiation.**

$$x^3 - y^3 = 3xy$$

Differentiating the equation with respect to x we receive

$$3x^2 - 3y^2\,\frac{dy}{dx} = 3y + 3x\,\frac{dy}{dx}$$

$$3x^2 - 3y = 3y^2\,\frac{dy}{dx} + 3x\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 - y}{y^2 + x}$$

8. { 15 points } **Use implicit differentiation to find the tangent line to the curve**

$$y = x \tan\left(\frac{\pi y}{2}\right) , \qquad x > 0,\ y > 0$$

at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Differentiating the equation with respect to x we receive

$$\frac{dy}{dx} = \tan\left(\frac{\pi y}{2}\right) + x\sec^2\left(\frac{\pi y}{2}\right)\left(\frac{\pi}{2}\,\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} - \frac{x\pi}{2}\sec^2\left(\frac{\pi y}{2}\right)\,\frac{dy}{dx} = \tan\left(\frac{\pi y}{2}\right)$$

Substituting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we receive

$$\left(1 - \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)\right)\,\frac{dy}{dx} = \tan\left(\frac{\pi}{4}\right)$$

and therefore

$$m = \frac{dy}{dx} = \frac{1}{1 - \frac{\pi}{4}\left(\sqrt{2}\right)^2} = \frac{1}{1 - \frac{\pi}{2}}$$

Thus, the tangent line is $\quad y - \frac{1}{2} = \frac{1}{1 - \frac{\pi}{2}}\left(x - \frac{1}{2}\right).$

9. { 15 points } **Find $\frac{dy}{dx}$ using logarithmic differentiation.**

$$y = \frac{\cos^4 x}{\sqrt{x^2+1}}$$

We have that $\ln y = \ln\left(\frac{\cos^4 x}{\sqrt{x^2+1}}\right) = 4\ln(\cos x) - \frac{1}{2}\ln(x^2+1)$ **and therefore**

$$\frac{1}{y}\,\frac{dy}{dx} = \frac{d}{dx}\,\ln y = 4\,\frac{1}{\cos x}\,(-\sin x) - \frac{1}{2}\,\frac{1}{x^2+1}\,(2x) = -4\tan x - \frac{x}{x^2+1}$$

Thus,

$$\frac{dy}{dx} = y\left(-4\tan x - \frac{x}{x^2+1}\right) = -\,\frac{\cos^4 x}{\sqrt{x^2+1}}\left(4\tan x + \frac{x}{x^2+1}\right).$$

10. { 15 points } **Find the limit.**

$$\lim_{x\to 1}\,\frac{\ln x}{x^3-1}$$

Since the limit is an indefinite form of the type $\frac{0}{0}$ we can apply the L'Hopital's Rule to obtain

$$\lim_{x\to 1}\,\frac{\ln x}{x^3-1} = \lim_{x\to 1}\,\frac{\frac{1}{x}}{3x^2} = \frac{1}{3}$$

First Bonus Problem. { 20 points }
The hypotenuse of a right triangle is growing at a constant rate of 3 centimeters per second and one leg is decreasing at a constant rate of 2 centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 5 centimeters?

We denote the hypotenuse by $h = h(t)$, the first leg by $\ell = \ell(t)$, and the angle between the hypotenuse and the other leg by $\alpha = \alpha(t)$. Then we have that at the considered instant $\frac{dh}{dt} = 3$, $\frac{d\ell}{dt} = -2$, $\ell = 5$, and $h = \sqrt{5^2 + 5^2} = 5\sqrt{2}$. Using that $\sin\alpha = \frac{\ell}{h}$ we receive $\alpha = \sin^{-1}\left(\frac{\ell}{h}\right)$ and therefore

$$\frac{d\alpha}{dt} = \frac{1}{\sqrt{1 - \left(\frac{\ell}{h}\right)^2}} \; \frac{\frac{d\ell}{dt}\, h \; - \; \ell\, \frac{dh}{dt}}{h^2} = \frac{1}{\sqrt{1 - \left(\frac{5}{5\sqrt{2}}\right)^2}} \; \frac{(-2)5\sqrt{2} - 5(3)}{\left(5\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{1 - \frac{1}{2}}} \; \frac{-10\sqrt{2} - 15}{50} = \sqrt{2}\left(- \frac{\sqrt{2}}{5} - \frac{3}{10}\right) = - \frac{2}{5} - \frac{3\sqrt{2}}{10}$$

Second Bonus Problem. { 20 points } **Find the limit.**

$$\lim_{x \to 0} \sqrt{\frac{x^2 e^x}{\sin(3x^2)}}$$

Using the properties of the limit and then the L'Hopital's Rule for indefinite forms of the type $\frac{0}{0}$ we obtain

$$\lim_{x \to 0} \sqrt{\frac{x^2 e^x}{\sin(3x^2)}} = \sqrt{\lim_{x \to 0} \frac{x^2 e^x}{\sin(3x^2)}} = \sqrt{\lim_{x \to 0} \frac{2xe^x + x^2 e^x}{\cos(3x^2)\;(6x)}}$$

$$= \sqrt{\lim_{x \to 0} \frac{2e^x + xe^x}{\cos(3x^2)\;(6)}} = \sqrt{\frac{2 + 0}{(1)(6)}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$