SOLUTIONS for Exam # 2

1. { 10 points } Find $f'(x)$ and $f''(x)$.

$$f(x) = x^7 - 4\sqrt{x} + \frac{1}{2} x^{-2}$$

$$f'(x) = 7x^6 - 2x^{-\frac{1}{2}} - x^{-3}$$

$$f''(x) = 42x^5 + x^{-\frac{3}{2}} + 3x^{-4}$$

2. { 10 points } Find $f'(x)$.

$$f(x) = \frac{e^{-2x}}{x^2 + \cos^5 x}$$

$$f'(x) = \frac{-2e^{-2x}(x^2 + \cos^5 x) - e^{-2x}(2x + 5\cos^4 x(-\sin x))}{(x^2 + \cos^5 x)^2}$$

$$= \frac{e^{-2x}(-2x^2 - 2\cos^5 x - 2x + 5\cos^4 x\sin x)}{(x^2 + \cos^5 x)^2}$$

3. { 10 points } Find $\frac{dy}{dx}$.

$$y = xe^{\sqrt{x}} + \ln x$$

$$\frac{dy}{dx} = e^{\sqrt{x}} + xe^{\sqrt{x}} \left( \frac{1}{2} x^{-\frac{1}{2}} \right) + \frac{1}{x} = e^{\sqrt{x}} + \frac{\sqrt{x}}{2} e^{\sqrt{x}} + \frac{1}{x}$$

4. { 10 points } Find $\frac{dy}{dx}$.

$$y = \sqrt{\cos^{-1}(x^3)}$$

$$\frac{dy}{dx} = \frac{1}{2} \left( \cos^{-1}(x^3) \right)^{-\frac{1}{2}} \left( -\frac{1}{\sqrt{1-(x^3)^2}} \right) (3x^2) = \frac{-3x^2}{2\sqrt{\cos^{-1}(x^3)} \sqrt{1-x^6}}$$
5. { 10 points } Find the values of \( x \) at which the curve \( y = f(x) \) has a horizontal tangent line, if \( f(x) = (2x + 5)^2(x - 1)^6 \).

Horizontal line means slope \( m = 0 \). Hence, we have to find all points \( x \) for which \( f'(x) = 0 \).

\[
\begin{align*}
f'(x) &= 2(2x + 5)(2) (x - 1)^6 + (2x + 5)^2 6(x - 1)^5 \\
&= (2x + 5)(x - 1)^5(4(x - 1) + 6(2x + 5)) = (2x + 5)(x - 1)^5(16x + 26).
\end{align*}
\]

Thus, \( x = -\frac{2}{5}, \ x = 1, \) and \( x = -\frac{26}{16} = -\frac{13}{8} \) are the points for which the tangent is horizontal.

6. { 15 points } Complete each part for the function \( f(x) = x^2 - 4x \).

(a) Find the slope of the tangent line to the graph of \( f \) at a general \( x \)-value.

\[
m = f'(x) = 2x - 4
\]

(b) Find the tangent line to the graph of \( f \) at \( x = 1 \).

The equation of the tangent line is \( y - y_0 = m(x - x_0) \) where \( x_0 = 1, \ y_0 = f(x_0) = 1 - 4 = -3, \) and \( m = f'(x_0) = 2 - 4 = -2 \).

Substituting these values we receive \( y - (-3) = -2(x - 1) \) and therefore \( y = -2x - 1 \) is the equation of the tangent line.

7. { 15 points } Find \( \frac{dy}{dx} \) by implicit differentiation.

\[
x^3 - y^3 = 3xy
\]

Differentiating the equation with respect to \( x \) we receive

\[
\begin{align*}
3x^2 - 3y^2 \frac{dy}{dx} &= 3y + 3x \frac{dy}{dx} \\
3x^2 - 3y &= 3y^2 \frac{dy}{dx} + 3x \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{x^2 - y}{y^2 + x}
\end{align*}
\]

8. { 15 points } Use implicit differentiation to find the tangent line to the curve

\[
y = x \tan \left( \frac{\pi y}{2} \right), \quad x > 0, \ y > 0
\]

at the point \( \left( \frac{1}{2}, \frac{1}{2} \right) \).
Differentiating the equation with respect to $x$ we receive

\[
\frac{dy}{dx} = \tan\left(\frac{\pi y}{2}\right) + x \sec^2\left(\frac{\pi y}{2}\right) \left(\frac{\pi}{2} \frac{dy}{dx}\right)
\]

\[
\frac{dy}{dx} - \frac{x\pi}{2} \sec^2\left(\frac{\pi y}{2}\right) \frac{dy}{dx} = \tan\left(\frac{\pi y}{2}\right)
\]

Substituting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we receive

\[
\left(1 - \frac{\pi}{4} \sec^2\left(\frac{\pi}{4}\right)\right) \frac{dy}{dx} = \tan\left(\frac{\pi}{4}\right)
\]

and therefore

\[
m = \frac{dy}{dx} = \frac{1}{1 - \frac{\pi}{4} (\sqrt{2})^2} = \frac{1}{1 - \frac{\pi}{2}}
\]

Thus, the tangent line is

\[
y - \frac{1}{2} = \frac{1}{1 - \frac{\pi}{2}} \left(x - \frac{1}{2}\right).
\]

9. { 15 points} Find $\frac{dy}{dx}$ using logarithmic differentiation.

\[
y = \frac{\cos^4 x}{\sqrt{x^2 + 1}}
\]

We have that \[\ln y = \ln\left(\frac{\cos^4 x}{\sqrt{x^2 + 1}}\right) = 4 \ln(\cos x) - \frac{1}{2} \ln(x^2 + 1)\] and therefore

\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \ln y = 4 \frac{1}{\cos x} (-\sin x) - \frac{1}{2} \frac{1}{x^2 + 1} (2x) = -4 \tan x - \frac{x}{x^2 + 1}
\]

Thus,

\[
\frac{dy}{dx} = y \left(-4 \tan x - \frac{x}{x^2 + 1}\right) = - \frac{\cos^4 x}{\sqrt{x^2 + 1}} \left(4 \tan x + \frac{x}{x^2 + 1}\right).
\]

10. { 15 points} Find the limit.

\[
\lim_{x \to 1} \frac{\ln x}{x^3 - 1}
\]

Since the limit is an indefinite form of the type $\frac{0}{0}$ we can apply the L’Hôpital’s Rule to obtain

\[
\lim_{x \to 1} \frac{\ln x}{x^3 - 1} = \lim_{x \to 1} \frac{\frac{1}{x}}{3x^2} = \frac{1}{3}
\]
First Bonus Problem. \{ 20 points \}
The hypotenuse of a right triangle is growing at a constant rate of \( 3 \) centimeters per second and one leg is decreasing at a constant rate of \( 2 \) centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are \( 5 \) centimeters?

We denote the hypotenuse by \( h = h(t) \), the first leg by \( \ell = \ell(t) \), and the angle between the hypotenuse and the other leg by \( \alpha = \alpha(t) \). Then we have that at the considered instant \( \frac{dh}{dt} = 3 \), \( \frac{d\ell}{dt} = -2 \), \( \ell = 5 \), and \( h = \sqrt{5^2 + 5^2} = 5\sqrt{2} \). Using that \( \sin \alpha = \frac{\ell}{h} \) we receive \( \alpha = \sin^{-1} \left( \frac{\ell}{h} \right) \) and therefore

\[
\frac{d\alpha}{dt} = \frac{1}{\sqrt{1 - \left( \frac{\ell}{h} \right)^2}} \frac{\frac{d\ell}{dt} h - \ell \frac{dh}{dt}}{h^2} = \frac{1}{\sqrt{1 - \left( \frac{5}{5\sqrt{2}} \right)^2}} \frac{(-2)5\sqrt{2} - 5(3)}{(5\sqrt{2})^2} \\
= \frac{1}{\sqrt{1 - \frac{1}{2}}} \frac{-10\sqrt{2} - 15}{50} = \sqrt{2} \left( -\frac{\sqrt{2}}{5} - \frac{3}{10} \right) = -\frac{2}{5} - \frac{3\sqrt{2}}{10}
\]

Second Bonus Problem. \{ 20 points \} Find the limit.

\[
\lim_{x \to 0} \sqrt{\frac{x^2e^x}{\sin(3x^2)}}
\]

Using the properties of the limit and then the L’Hopital’s Rule for indefinite forms of the type \( \frac{0}{0} \) we obtain

\[
\lim_{x \to 0} \sqrt{\frac{x^2e^x}{\sin(3x^2)}} = \sqrt{\lim_{x \to 0} \frac{x^2e^x}{\sin(3x^2)}} = \sqrt{\lim_{x \to 0} \frac{2xe^x + x^2e^x}{\cos(3x^2) (6x)}} \\
= \sqrt{\lim_{x \to 0} \frac{2e^x + xe^x}{\cos(3x^2) (6)}} = \sqrt{\frac{2 + 0}{(1)(6)}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
\]