SOLUTIONS for Exam # 1

1. { 10 points } **Complete the following table** *without explanations.*

x	-2	-1	0	1	2
$f(x)$	-1	2	0	-2	1
$g(x)$	2	0	-1	2	-1
$(f \circ g)(x)$	**1**	**0**	**2**	**1**	**2**
$(g \circ f)(x)$	**0**	**-1**	**-1**	**2**	**2**

2. { 10 points } **Classify the functions as even, odd, or neither.** *Check the appropriate boxes without explanations.*

function	odd	even	neither
$f(x)$ from problem 1.	✓	☐	☐
$g(x)$ from problem 1.	☐	☐	✓
$\cos^3(x)$	☐	✓	☐
$x - x^3$	✓	☐	☐
$x^2 + \sin x$	☐	☐	✓

3. { 5 points } **State the domain of the function** $f(x) = \frac{x}{\sqrt{x-2}}$.

$\mathcal{D} = (2, \infty)$ **or** $x > 2$

We have that $x - 2 \geq 0$ **for the square root and also the denominator should not be** 0**, i.e.** $x - 2 \neq 0$.

4. { 10 points } **Given the functions** $f(x) = \frac{x}{3x-1}$ **and** $g(x) = \frac{1}{x^2-1}$, **find the composition** $f \circ g$ **and state its domain.**

$$(f \circ g)(x) = \frac{\frac{1}{x^2-1}}{3\left(\frac{1}{x^2-1}\right)-1} = \frac{1}{3-(x^2-1)} = \frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)}.$$

$\mathcal{D} = \{$ all real $x \neq \pm 1, \pm 2\}$.

5. { 10 points } **In each part, find the inverse** $f^{-1}(x)$ **if the inverse exists.**

(a) $f(x) = \cos(2x)$, $0 \leq x \leq \frac{\pi}{2}$

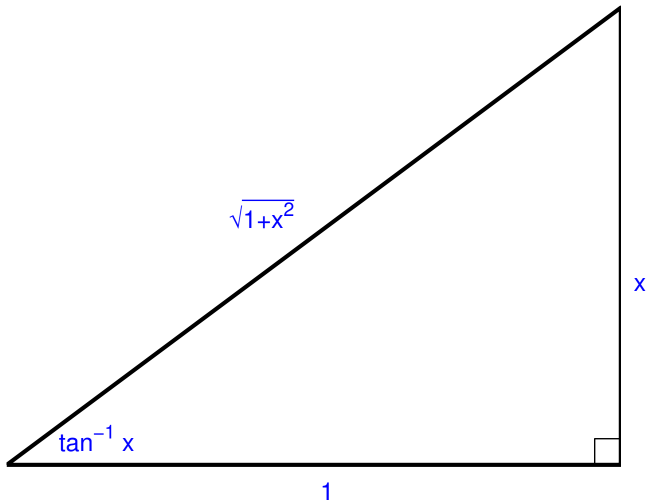
We have that $0 \leq 2x \leq \pi$ **for which interval** $\cos^{-1}$ **is defined. Thus,** $y = \cos(2x)$ **is equivalent to** $\cos^{-1} y = 2x$ **and therefore** $x = \frac{1}{2} \cos^{-1} y$**. So,** $f^{-1}(x) = \frac{1}{2} \cos^{-1} x$ **.**

(b) $f(x) = \sin(2x)$, $0 \leq x \leq \frac{\pi}{2}$

The function $\sin(2x)$ **is not one-to-one for** $0 \leq 2x \leq \pi$ **(we have that** $\sin(2x) = \sin(\pi - 2x)$ **) and therefore there is no inverse to** f**.**

6. { 10 points } **Complete the identity using the triangle method.**

$\cos\left(\tan^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$



7. { 10 points } **Solve for** x **without using a calculating utility.**

$$2\ln\left(\frac{-1}{x}\right) + \ln(2x^4) = \ln 8$$

We have that $2\ln\left(\frac{-1}{x}\right) + \ln(2x^4) = \ln\left(\left(\frac{-1}{x}\right)^2 (2x^4)\right) = \ln\left(2x^2\right)$ **.**

So, $\ln\left(2x^2\right) = \ln 8$ **and therefore** $2x^2 = 8$ **which gives** $x^2 = 4$ **and** $x = \pm 2.$ **However, the domain for** $\ln\left(\frac{-1}{x}\right)$ **is** $x < 0$ **and therefore** $x = 2$ **should be excluded.**

Thus, the solution is $x = -2.$

8. { 12 points } **Find the limit.**

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x}\,\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x\to 0} \frac{(x^2+9)-3^2}{x\sqrt{x^2+9}+3}$$

$$= \lim_{x\to 0} \frac{x^2}{x\sqrt{x^2+9}+3} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+9}+3}$$

$$= \frac{0}{\sqrt{0+9}+3} = 0$$

9. { 12 points } **Find the limit.**

$$\lim_{x\to+\infty} \left(1-\frac{5}{x}\right)^{x}$$

$$= \lim_{x\to+\infty} \left(1-\frac{5}{x}\right)^{-\frac{x}{5}(-5)} = \left(\lim_{x\to+\infty} \left(1-\frac{5}{x}\right)^{-\frac{x}{5}}\right)^{-5} = e^{-5} = \frac{1}{e^5}\ .$$

10. { 12 points } **Find the limit.**

$$\lim_{x\to+\infty} \frac{(3-2x)^3}{x(7+x+9x^2)}$$

$$= \lim_{x\to+\infty} \frac{(3-2x)^3 \ \frac{1}{x^3}}{x(7+x+9x^2)\ \frac{1}{x^3}} = \lim_{x\to+\infty} \frac{\left(\frac{3-2x}{x}\right)^3}{\frac{7+x+9x^2}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{\left(\frac{3}{x}-2\right)^3}{\frac{7}{x^2}+\frac{1}{x}+9} = \frac{(0-2)^3}{0+0+9} = -\frac{8}{9}\ .$$

11. { 12 points } **Find the limit.**

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin\left(\frac{x}{2}\right)}$$

$$= \lim_{x\to 0} \left(\frac{\tan(3x)}{3x}\right)\left(\frac{\frac{x}{2}}{\sin\left(\frac{x}{2}\right)}\right)\left(\frac{3x}{\frac{x}{2}}\right)$$

$$= \left(\lim_{x\to 0} \frac{\sin(3x)}{3x\ \cos(3x)}\right)\left(\lim_{x\to 0} \frac{\frac{x}{2}}{\sin\left(\frac{x}{2}\right)}\right)\left(\lim_{x\to 0} \frac{3}{\frac{1}{2}}\right) = (1)(1)(6) = 6\ .$$

12. { 12 points } **A positive number** $\varepsilon = 0.01$ **and a limit** $L = 4$ **for the function** $f(x) = \frac{9x^2-4}{3x-2}$ **at** $a = \frac{2}{3}$ **are given:**

$$\lim_{x\to\frac{2}{3}} \frac{9x^2-4}{3x-2} = 4\ .$$

Find a number δ **such that** $|f(x)-L| < \varepsilon$ **if** $0 < |x-a| < \delta$ **.**

We have $|f(x)-L| = \left|\frac{9x^2-4}{3x-2} - 4\right| = \left|\frac{(3x-2)(3x+2)}{3x-2} - 4\right| = |3x+2-4|$

$= |3x-2| = 3\left|x-\frac{2}{3}\right| < 3\delta$ **for** $0 < |x-a| = \left|x-\frac{2}{3}\right| < \delta$ **.**

So, if we set δ **such that** $3\delta \le \varepsilon = 0.01$ **, for example** $\delta = 0.0033$ **,**

we shall have that $|f(x)-4| < 0.01$ **if** $0 < \left|x-\frac{2}{3}\right| < \delta$ **.**

Bonus Problem. { 20 points } **Find the limit.**

$$\lim_{x\to 0}\left(\frac{1-\cos(2x)}{x\sin(2x)} + x\ \sin\left(1/x\right)\right)$$

$$= \quad \left(\lim_{x\to 0}\ \frac{1-\cos(2x)}{x\sin(2x)}\right) \quad + \quad \left(\lim_{x\to 0}\ x\ \sin\left(1/x\right)\right) \ .$$

For the first limit we have

$$\lim_{x\to 0}\ \frac{1-\cos(2x)}{x\sin(2x)} \quad = \quad \lim_{x\to 0}\ \frac{1-\cos(2x)}{x\sin(2x)}\ \frac{1+\cos(2x)}{1+\cos(2x)}$$

$$= \quad \lim_{x\to 0}\ \frac{1-\cos^2(2x)}{x\sin(2x)(1+\cos(2x))} \quad = \quad \lim_{x\to 0}\ \frac{\sin^2(2x)}{x\sin(2x)(1+\cos(2x))}$$

$$= \quad \lim_{x\to 0}\ \frac{\sin(2x)}{x(1+\cos(2x))} \quad = \quad \lim_{x\to 0}\ \frac{\sin(2x)}{2x}\ \frac{2}{1+\cos(2x)} \quad = \quad (1)\ \frac{2}{1+1} \quad = \quad 1\ .$$

For the second limit we shall use the inequalities $-1 \le \sin\left(1/x\right) \le 1$ **and therefore** $-|x| \le x\ \sin\left(1/x\right) \le |x|$ **.**

Since $|x| \to 0$ **as** $x \to 0$ **by the Squeezing Theorem we have that** $\lim_{x\to 0}\ \left(x\ \sin\left(1/x\right)\right) = 0$ **.**

Finally,

$$\lim_{x\to 0}\left(\frac{1-\cos(2x)}{x\sin(2x)} + x\ \sin\left(1/x\right)\right) \quad = \quad 1\ +\ 0 \quad = \quad 1\ .$$