ON THE "MULTIPLE OF THE INCLUSION PLUS COMPACT" PROBLEM

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Abstract. The "multiple of the inclusion plus compact problem" which was posed by T.W. Gowers in 1996 and Th. Schlumprecht in 2003, asks whether for every infinite dimensional Banach space $X$ there exists a closed subspace $Y$ of $X$ and a bounded linear operator from $Y$ to $X$ which is not a compact perturbation of a multiple of the inclusion map from $Y$ to $X$. We give sufficient conditions on the spreading models of seminormalized basic sequences of a Banach space $X$ which guarantee that the "multiple of the inclusion plus compact" problem has an affirmative answer for $X$. We give an example of a Hereditarily Indecomposable Banach space where our results apply. For the proof of our main result we use an extension of E. Odell’s Schreier unconditionality result for arrays.

0. Introduction

A long-standing open famous question of J. Lindenstrauss asks whether on every infinite dimensional Banach space $X$ there exists a (linear bounded) operator from $X$ to $X$ which is not a compact perturbation of a multiple of the identity operator on $X$. A weaker question was asked by T.W. Gowers in 1996 [10] and by Th. Schlumprecht in 2003 [20]: does every infinite dimensional Banach space $X$ admit a (closed) subspace $Y$ and an operator from $Y$ to $X$ which is not a compact perturbation of a multiple of the inclusion operator from $Y$ to $X$. We refer to this question as the "multiple of the inclusion plus compact" problem. The main result of this paper gives sufficient conditions on the spreading models of seminormalized basic sequences of a Banach space $X$ which ensure that there exists a subspace $Y$ of $X$ having a basis and an operator from $Y$ to $X$ which is not a compact perturbation of the inclusion map from $Y$ to $X$.

If $X$ and $Y$ are Banach spaces, let $\mathcal{L}(X,Y)$ (respectively $\mathcal{K}(X,Y)$), denote the set of all (respectively compact) operators from $Y$ to $X$. If $Y$ is a subspace of $X$ let $i_{Y \to X}$ denote the inclusion map from $Y$ to $X$. If $Y$ is a subspace of $X$ and $T \in \mathcal{L}(Y,X)$ then the statement $T \not\in Ci_{Y \to X} + \mathcal{K}(Y,X)$, means that $T$ cannot be written as a compact perturbation of a multiple of the inclusion map from $Y$ to $X$. We say that the "multiple of the inclusion plus compact" problem has an affirmative answer on a Banach space $X$ if there exists a subspace $Y$ of $X$ such that $T \not\in Ci_{Y \to X} + \mathcal{K}(Y,X)$. If $(x_n)_n$ is a basic sequence in a Banach space, let $[(x_n)_n]$ denote the closed linear span of the sequence $(x_n)_n$.

Note that if a Banach space $X$ contains an unconditional basic sequence $(x_n)_n$ then the operator $T \in \mathcal{L}([(x_n)_n],X)$ defined by $T(x_n) = (-1)^nx_n$ does not belong to $Ci_{Y \to X} + \mathcal{K}(Y,X)$.

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Thus if a Banach space $X$ contains an unconditional basic sequence then the “multiple of the inclusion plus compact” problem has an affirmative answer for $X$. It is not known whether the above question of Lindenstrauss has an affirmative answer in this case. Hence for the “multiple of the inclusion plus compact” problem we restrict our attention to Banach spaces with no unconditional basic sequences. Recall that by the Gowers’ dichotomy [9] every Banach space contains an unconditional basic sequence or a hereditarily indecomposable (HI) subspace. Recall that a Banach space $X$ is called HI if no infinite dimensional closed subspace $Y$ of $X$ contains a complemented subspace $Z$ which is of both infinite dimension and infinite codimension in $Y$ [11]. Therefore for the “multiple of the inclusion plus compact problem” we only examine HI saturated Banach spaces.

The “multiple of the inclusion plus compact” problem was first studied by Gowers [10] where he proved that it has an affirmative answer for the HI Banach space $GM$ which was constructed by Gowers and B. Maurey [11]. Moreover, Gowers conjectured that this problem has an affirmative answer for all reflexive Banach spaces.

Subsequently, Schlumprecht [20] studied the “multiple of the inclusion plus compact” problem and gave sufficient conditions on a Banach space $X$ so that this problem has an affirmative answer on $X$. One of the main results in [20] gives sufficient conditions on the spreading models of weakly null sequences of an infinite dimensional Banach space $X$ which ensure that the “multiple of the inclusion plus compact” problem has an affirmative answer on $X$. Recall that [4, 5, 6] for every seminormalized basic sequence $(y_n)$ in a Banach space $X$ and for every $(\varepsilon_n) \searrow 0$ there exists a subsequence $(x_{n_i})$ of $(y_n)$ and a seminormalized basic sequence $(\tilde{x}_n)$ (not necessarily in $X$) such that for all $n \in \mathbb{N}$, scalars $(a_i)_{i=1}^n$ with $|a_i| \leq 1$ and $n \leq k_1 < \cdots < k_n$,

$$\left| \left\| \sum_{i=1}^n a_{k_i} x_{k_i} \right\| - \left\| \sum_{i=1}^n a_i \tilde{x}_i \right\| \right| < \varepsilon_n.$$  

The sequence $(\tilde{x}_n)$ is called a spreading model of $(x_n)$. If $X$ is a Banach space then $SP_w(X)$ will denote the set of spreading models of seminormalized weakly null basic sequences of $X$. If $(x_n)$ is a seminormalized weakly null basic sequence then $(\tilde{x}_n)$ is an unconditional basic sequence. Thus if $X$ is an HI Banach space and $(x_n)$ is a seminormalized basic sequence in $X$ with spreading model $(\tilde{x}_n)$, then it may be easier to study $(\tilde{x}_n)$ than to study $(x_n)$ itself. Schlumprecht [20] introduced the following crucial notion (without assigning a name):

**Definition 0.1.** Let $(x_n)$ and $(z_n)$ be two seminormalized basic sequences (not necessarily in the same Banach space). For $\varepsilon > 0$ define

$$(1) \quad \Delta_{(z_n),(x_n)}(\varepsilon) := \sup\{ \| \sum a_i x_i \| : (a_i) \in c_{00}, |a_i| \leq \varepsilon \text{ and } \| \sum a_i z_i \| \leq 1 \},$$

where $c_{00}$ denotes the linear space of finitely supported scalar sequences. We say that $(z_n)$ dominates $(x_n)$ on small coefficients, (denoted by $(x_n) \ll (z_n)$ and abbreviated as “$(z_n)$ s.c. dominates $(x_n)$”), if

$$(2) \quad \lim_{\varepsilon \searrow 0} \Delta_{(z_n),(x_n)}(\varepsilon) = 0.$$

Obviously, if $(z_i)$ and $(x_i)$ are seminormalized basic sequences in some Banach spaces and $(z_i) \gg (x_i)$ then

$$(3) \quad \liminf_{\varepsilon \searrow 0} \{ \| \sum a_i z_i \| : |a_i| \leq \varepsilon \text{ and } \| \sum a_i x_i \| = 1 \} = \infty.$$
where we assume that \( \inf \emptyset = \infty \).

Another important notion that was introduced by Schlumprecht [20] was the following property which is called “Property P1” in the present article.

**Definition 0.2.** A seminormalized basic sequence \((z_i)\) has Property P1 if

\[
\liminf_{n \to \infty} \inf_{A \subset \mathbb{N}, |A|=n} \| \sum_{i \in A} z_i \| = \infty.
\]

One of the main results in [20] is the following powerful result:

**Theorem 0.3.** Let \(X\) be an infinite dimensional Banach space. Let \((x_n)\) and \((z_n)\) be normalized weakly null basic sequences in \(X\) having spreading models \((\tilde{x}_n)\) and \((\tilde{z}_n)\) respectively, such that \((\tilde{x}_n) \ll (\tilde{z}_n)\) and \((\tilde{z}_n)\) has Property P1. Then the “multiple of the inclusion plus compact” problem has an affirmative answer on \(X\).

Another main result in [20] is its Theorem 1.4. The idea of that important result is that an ordinal index is assigned to every normalized basic sequence of a Banach space, taking values at most equal to the first uncountable ordinal \(\omega_1\). Heuristically speaking, this index measures how close is the basic sequence that we examine to the unit vector basis of \(\ell_1\). Then an index is assigned to a Banach space as the supremum of the indices of its normalized weakly null basic sequences. Roughly speaking, [20, Theorem 1.4] states that if the index of the Banach space \(X\) is larger than the index of one weakly null normalized basic sequence in \(X\), then the “multiple of the inclusion plus compact” problem has an affirmative answer on \(X\).

Another sufficient condition for the “multiple of the inclusion plus compact” problem to have an affirmative solution on a Banach space was given by the first named author, E. Odell, Th. Schlumprecht and N. Tomczak-Jaegermann [2]. This is a sufficient condition on the spreading models of normalized weakly null basic sequences of a Banach space \(X\). In order to state the mentioned result of [2], we need some more definitions that we give now.

A normalized basic sequence \((x_n)\) is called 1-subsymmetric if it is 1-equivalent to all of its subsequences.

**Definition 0.4.** Let \((x_n)\) be a 1-subsymmetric basic sequence in some Banach space. The Krivine set of \((x_n)\) is defined to be the set of all \(p\)'s in \([1, \infty]\) with the following property. For all \(\varepsilon > 0\) and \(N \in \mathbb{N}\) there exists \(m \in \mathbb{N}\) and scalars \((\lambda_k)_{k=1}^m\) such that for all scalars \((a_n)_{n=1}^N\),

\[
\frac{1}{1 + \varepsilon} \| (a_n)_{n=1}^N \|_p \leq \| \sum_{n=1}^N a_n y_n \| \leq (1 + \varepsilon) \| (a_n)_{n=1}^N \|_p
\]

where

\[
y_n = \sum_{k=1}^m \lambda_k x_{(n-1)m+k} \text{ for } n = 1, \ldots, N
\]

and \(\| \cdot \|_p\) denotes the norm of the space \(\ell_p\).

The Krivine’s theorem as it was proved by H. Rosenthal [16] and H. Lemberg [12] states that if \((x_n)\) is a 1-subsymmetric basic sequence then the Krivine set of \((x_n)\) is non-empty. In particular, if \((x_n)\) is a seminormalized basic sequence in a Banach space having spreading model \((\tilde{x}_n)\) then \((\tilde{x}_n)\) is 1-subsymmetric, hence the Krivine set of \((\tilde{x}_n)\) is non-empty. The following result was proved in [2]:

3
**Theorem 0.5.** Let $X$ be a Banach space. Assume that there exist normalized weakly null basic sequences $(x_n), (z_n)$ in $X$ such that $(x_n)$ has a spreading model $(\tilde{x}_n)$ which is not equivalent to the unit vector basis of $\ell_1$ and $(z_n)$ has a spreading model $(\tilde{z}_n)$ such that 1 belongs to the Krivine set of $(\tilde{z}_n)$. Then the “multiple of the inclusion plus compact problem” has an affirmative answer on $X$.

This result strengthens the result of Gowers [10] since the Banach space $GM$ satisfies the condition of Theorem 0.5.

In order to state the main result of our paper, we need to introduce a property which is closely related to the Property P1 and it is called Property P2 in the present article. The Property P2 appears without a name in [2].

**Definition 0.6.** A seminormalized basic sequence $(x_n)$ has Property P2 if for all $\rho > 0$ there exists an $M = M(\rho) \in \mathbb{N}$, such that if $\| \sum a_i z_i \| = 1$ then $|\{i : |a_i| \geq \rho\}| \leq M$.

One of the main results of our paper is the following:

**Theorem 0.7.** Let $X$ be a Banach space containing seminormalized basic sequences $(x_i)_i$ and $(x^n_i)_i$ for all $n \in \mathbb{N}$, such that $0 < \inf_n, i \|x^n_i\| \leq \sup_n, i \|x^n_i\| < \infty$. Let $(z_i)_i$ be a basic sequence not necessarily in $X$. Assume that $(z_i)$ satisfies:

\[ (5) \quad \text{The sequence } (x_i)_i \text{ has a spreading model } (\tilde{x}_i)_i^{\infty} \text{ such that } (\tilde{x}_i)_{i \in \mathbb{N}} \ll (z_i)_{i \in \mathbb{N}}. \]

Assume that for all $n \in \mathbb{N}$ the sequence $(x^n_i)_i$ satisfies:

\[ (6) \quad \text{The sequence } (x^n_i)_i \text{ has a spreading model } (\tilde{x}_i^n)_i \text{ such that } (z_i)_i^n \text{ is } C\text{-dominated by } (\tilde{x}_i^n)_i \text{ for some } C \text{ independent of } n. \]

Assume that the sequence $(z_i)$ has a spreading model which has Property P2. Then there exists a subspace $Y$ of $X$ which has a basis and an operator $T \in \mathcal{L}(Y, X)$ which is not a compact perturbation of a multiple of the inclusion map.

Theorem 0.7 obviously implies Theorem 0.3. Indeed, if the assumptions of Theorem 0.3 apply for a Banach space $X$, then let the sequence “$(z_i)$” of Theorem 0.7 to be the sequence $(\tilde{z}_n)$ (which appears in the assumptions of Theorem 0.3) and the sequences $(x^n_i)_i$ of Theorem 0.7 to be all equal to the sequence $(z_n)$ (which appears in the assumptions of Theorem 0.3). Notice that since $(z_n)$ is weakly null, we have that $(\tilde{z}_n)$ is unconditional by [5], [6] and it is trivial to verify that any seminormalized unconditional basic sequence with Property P1 must have Property P2. On the other hand we note that Theorem 0.7 can be proved using Theorem 0.3. The proof involves repetition of the proof that we present (namely Lemmas 3.2 and 3.3) as well as the use of [2, Proposition 5.2(b) and 3.2].

Another easy corollary of Theorem 0.7 is obtained if we set $(z_i)$ to be the unit vector basis of $\ell_p$ for some fixed $p \in [1, \infty)$. Then we obtain the following result (we present its short proof in Section 3).

**Theorem 0.8.** Let $X$ be a Banach space. Assume that there exist seminormalized basic sequences $(x_i), (y_i)$ in $X$ such that $(x_i)$ has spreading model $(\tilde{x}_i)$ which is s.c. dominated by the unit vector basis of $\ell_p$, for some $p \in [1, \infty)$ and $(y_i)$ has spreading model $(\tilde{y}_i)$ such that $p$ belongs to the Krivine set of $(\tilde{y}_i)$. Then there exists a subspace $Y$ of $X$ with a basis and an operator $T \in \mathcal{L}(Y, X)$ which is not a compact perturbation of a multiple of the inclusion map.
By [2, Proposition 2.1] we know that a seminormalized subsymmetric basic sequence is not equivalent to the unit vector basis of \( \ell_1 \) if and only if it is s.c. dominated by the unit vector basis of \( \ell_1 \). Thus Theorem 0.8 for \( p = 1 \) implies Theorem 0.5.

In Section 1 we present an equivalent statement to the following question. Given a Banach space \( X \) does there exist a subspace \( Y \) having a basis and \( T \in \mathcal{L}(Y, X) \) such that \( T \not\in \mathcal{C}_{\ell_1 \to X} + \mathcal{K}(Y, X) \)? All of the above mentioned results in fact assert that this last problem has an affirmative answer on a Banach space \( X \) under the corresponding assumptions on \( X \) given by each result.

In Section 2 we extend the classical result of Odell on Schreier unconditionality. Recall that a finite subset \( F \) of \( \mathbb{N} \) is called a Schreier set if \( |F| \leq \min(F) \) (where \( |F| \) denotes the cardinality of \( F \)). A basic sequence \( (x_n) \) is defined to be \textit{Schreier unconditional} if there is a constant \( C > 0 \) such that for all scalars \( (a_i) \in c_{00} \) and for all Schreier sets \( F \) we have

\[
\| \sum_{i \in F} a_i e_i \| \leq C \| \sum a_i e_i \|.
\]

In this case \( (e_i) \) is called \( C \)-\textit{Schreier unconditional}. The important notion of Schreier unconditionality was introduced by E. Odell [15] and has inspired rich literature on the subject (see for example [3], [8]). Earlier very similar results can be found in [14, page 77] and [18, Theorem 2.1'].

**Theorem 0.9.** [15] Let \( (x_n) \) be a normalized weakly null sequence in a Banach space. Then for any \( \varepsilon > 0 \), \( (x_n) \) contains a \((2 + \varepsilon)\)-Schreier unconditional subsequence.

The main result of Section 2 is Theorem 2.9 where we extend Theorem 0.9 to arrays.

In Section 3 we prove Theorem 3.1 which also gives sufficient conditions on a Banach space \( X \) so that the “multiple of the inclusion plus compact” problem has an affirmative answer on \( X \). Then Theorem 3.1 is used in the proof of Theorem 0.7 which is further used in the proof of Theorem 0.8. The main result of Section 2, Theorem 2.9, plays an important role in the proof of Theorem 3.1. In Section 3 we also examine the relationship between the Properties P1 and P2.

As already mentioned we can restrict the question of the “multiple of the inclusion plus compact” problem to HI saturated spaces. So for a nontrivial application of the above results we need to look at the list of HI spaces. In his 2000 dissertation N. Dew [7] introduced a new HI space which we refer to as space \( D \). In Section 4 we examine some of the basic properties of \( D \) and we apply Theorem 0.8 to prove that the “multiple of the inclusion plus compact” problem has an affirmative answer in \( D \).

1. **AN ALMOST EQUIVALENT REFORMULATION OF THE “MULTIPLE OF THE INCLUSION PLUS COMPACT” PROBLEM**

A closely related problem to the “multiple of the inclusion plus compact” problem is the following

**Question 1.1.** Assume \( X \) is an infinite dimensional Banach space. Is there a closed subspace \( Y \) of \( X \) having a basis and an operator \( T \in \mathcal{L}(Y, X) \) so that \( T \not\in \mathcal{C}_{\ell_1 \to X} + \mathcal{K}(Y, X) \)?
Notice that Theorems 0.7 and 0.8 provide sufficient conditions for this question to have an affirmative answer. Moreover, the proofs of Theorems 0.3 and 0.5 also reveal that they provide an affirmative answer to this question.

Before presenting Proposition 1.3 which is the main result of the section, we start with the following remark which will be used in the proof of Proposition 1.3.

**Remark 1.2.** Let $X$ be a Banach space containing no unconditional basic sequence. Let $(x_n)$ be a seminormalized basic sequence in $X$ and $S \in \mathcal{L}((x_n), X)$ such that $(Sx_n)$ converges. Then there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that the restriction of $S$ on the span of $(x_{n_k} - x_{n_{2k-1}})$ is compact.

**Proof.** Since $X$ does not contain any unconditional basic sequence, no subsequence of $(x_n)$ is equivalent to the unit vector basis of $\ell_1$, hence by Rosenthal’s $\ell_1$ Theorem [17] there exists a subsequence $(x_{n_k})$ of $(x_n)$ which is weak Cauchy. Thus the sequence $(x_{n_k} - x_{n_{2k-1}})$ is seminormalized and weakly null. Since $(Sx_n)$ converges we have that $(S(x_{n_k} - x_{n_{2k-1}}))$ converges to zero. By passing to a further subsequence of $(x_n)$ and relabeling we may assume that $\sum \|S(x_{n_k} - x_{n_{2k-1}})\| < \infty$ which easily implies that the restriction of $S$ on the span of $(x_{n_k} - x_{n_{2k-1}})$ is compact. \qed

The next result gives an equivalent characterization of Question 1.1.

**Proposition 1.3.** Let $X$ be a Banach space. The following are equivalent.

(A) There exists a basic sequence $(x_n)$ in $X$ and an operator $T \in \mathcal{L}((x_n), X)$ such that $T \not\in \mathcal{C}_{i((x_n))} X + \mathcal{K}((x_n), X)$.

(B) There exists a seminormalized basic sequence $(x_n)$ in $X$ and a sequence $(y_n)$ in $X$ such that $(x_n)$ dominates $(y_n)$ and one of the following three happen.

(i) For all scalars $\lambda$, $(y_n - \lambda x_n)$ has no converging subsequence.

(ii) There exists a scalar $\lambda$, such that $(y_n - \lambda x_n)$ converges and there exists a bounded sequence $(z_n) \subseteq \text{span}(x_i)_{i=1}^\infty$ such that $(w_n - \lambda z_n)$ has no converging subsequence, where each $w_n \in \text{span}(y_i)_{i=1}^\infty$ has the same distribution with respect to $(y_i)_i$ as $z_n$ has with respect to $(x_i)_i$.

(iii) There exist scalars $\lambda_1 \neq \lambda_2$ and increasing sequences $(k^1_n)$, $(k^2_n)$ of $\mathbb{N}$ such that $(y_{k^1_n} - \lambda_i x_{k^1_n})$ converges for $i \in \{1, 2\}$.

**Proof.** Note that if the Banach space $X$ contains an unconditional basic sequence then (A) is satisfied as we noticed in the Introduction. Also, in that case, (B)(i) is satisfied (if we set $(x_n)$ to be a seminormalized unconditional basic sequence in $X$ and $(y_n)$ to be equal to $((-1)^n x_n)$). Thus we can restrict our attention to a Banach space $X$ containing no unconditional basic sequence (i.e. by Gowers’ dichotomy [9], $X$ is saturated with HI spaces).

To show (A) implies (B) let $T \in \mathcal{L}((x_n), X)$ for some seminormalized basic sequence $(x_n) \subseteq X$ where $T \not\in \mathcal{C}_{i((x_n))} X + \mathcal{K}((x_n), X)$. Then we see that $(x_n)$ dominates $(y_n) := (T(x_n))$. Then either $(y_n - \lambda x_n)$ has no converging subsequence for all scalars $\lambda$ (hence (i) holds) or there is a unique scalar $\lambda$ such that $(y_n - \lambda x_n)$ has a converging subsequence or there exist scalars $\lambda_1 \neq \lambda_2$ and increasing sequences $(k^1_n)$, $(k^2_n)$ of $\mathbb{N}$ such that $(y_{k^1_n} - \lambda_i x_{k^1_n})$ converges for $i \in \{1, 2\}$ (hence (iii) holds). Thus if (B) is not valid, then there exists a unique scalar $\lambda$ such that $(y_n - \lambda x_n)$ converges and for all bounded sequences $(z_n) \subseteq \text{span}(x_n)$ we have $(T(z_n) - \lambda z_n)$ converges. Thus $T - \lambda_1 j_{(x_n)} X$ is compact which is a contradiction.
To show (B) implies (A) we assume we have a pair of sequences \((x_n)\) and \((y_n)\) in \(X\) with
\((x_n)\) a seminormalized basic sequence, and \((x_n)\) dominating \((y_n)\). Define a bounded linear
operator \(T : ([x_n]) \to X\) by \(T(x_n) = y_n\). We show that in each case (i), (ii) and (iii) we have
\(T \not\in C_i([x_n]) → X + K(\{(x_n)\}, X)\).

**Case (i):** Assume \(T \in \lambda i([x_n]) → X + K(\{(x_n)\}, X)\) for some \(\lambda\). Then \(T - \lambda i([x_n]) → X\) is compact.
Notice that \((x_n)\) is bounded thus we have \(((T - \lambda i([x_n]) → X) x_n)_n = (y_n - \lambda x_n)_n\) has a convergent
subsequence. A contradiction to the assumption (i).

**Case (ii):** Assume that for some scalar \(\mu\) we have that \(T - \mu i([x_n]) → X \in K(\{(x_n)\}, X)\). On
the other hand, by Remark 1.2 applied to \(S := T - \lambda i([x_n]) → X\) and \((x_n)\) we have that there
exists a subsequence \((x_{nk})\) of \((x_n)\) such that the restriction of \(T - \lambda i([x_n]) → X\) on the span of
\((x_{n2k} - x_{n2k-1})_k\) is compact. Let \(Y\) denote the span of \((x_{n2k} - x_{n2k-1})_k\). Then the restriction
of the operator

\[
(\lambda - \mu)i([x_n]) → X = (T - \mu i([x_n]) → X) - (T - \lambda i([x_n]) → X)
\]
on \(Y\) is compact. Since \(Y\) is infinite dimensional we obtain that \(\lambda = \mu\). Hence \(T - \lambda i([x_n]) → X \in K(\{(x_n)\}, X)\). This contradicts the assumption (ii) that there exists a bounded sequence
\((z_n) \in \text{span}(x_i)_i\), such that \((w_n - \lambda z_n)_n = ((T - \lambda i([x_n]) → X)z_n)_n\) has no converging subsequence.

**Case (iii):** Assume that for some scalar \(\mu\) we have that \(T - \mu i([x_n]) → X \in K(\{(x_n)\}, X)\). Then
as in the proof of case (ii) we obtain that \(\mu = \lambda_1\) and \(\mu = \lambda_2\) contradicting the fact that
\(\lambda_1 \neq \lambda_2\).

\[\square\]

2. Extension of Odell’s Schreier Unconditionality

The main result of this section is Theorem 2.9 which is an extension of Theorem 0.9 to
an array of vectors of a Banach space such that each row is a seminormalized weakly null
sequence. Then Theorem 2.9 guarantees the existence of a subarray which preserves all the
rows of the original array and has a Schreier type of unconditionality. Theorem 2.9 will be
an important tool in the proof of the main result of this article (Theorem 0.7) in section 3.

We now define the notions of array, subarray and regular array in a Banach space. An
**array** in a Banach space \(X\) is a sequence of vectors in \((x_{i,j})_{i \in \mathbb{N}, j \in J_i} \subset X\) where \(J_i\) is an infinite
subsequence of \(\mathbb{N}\) for all \(i \in \mathbb{N}\), say \(J_i = \{j_{i,1} < j_{i,2} < \cdots\}\) and \((x_{i,j})_{i \in \mathbb{N}}\) is a seminormalized
weakly null sequence in \(X\) for all \(i \in \mathbb{N}\). Let \(<_{rl}\) denote the reverse lexicographical order on
\(\mathbb{N}^2\). Let \((x_{i,j})_{i \in \mathbb{N}, j \in J_i}\) be an array in a Banach space \(X\). A **subarray** of \((x_{i,j})_{i \in \mathbb{N}, j \in J_i}\) is an
array \((y_{i,\ell})_{i \in \mathbb{N}, \ell \in L_i}\) in \(X\) which satisfies the following two properties:

\[(7)\]

\[
\{y_{i,\ell} : \ell \in L_i\} \subseteq \{x_{i,j} : j \in J_i\} \text{ for all } i \in \mathbb{N}
\]

and

\[(8)\]

\[
\begin{align*}
\text{ if } J_i = \{j_{i,1} < j_{i,2} < \cdots\}, \text{ and } L_i = \{\ell_{i,1} < \ell_{i,2} < \cdots\} \text{ for all } i \in \mathbb{N} \text{ then there exists a } <_{rl} \text{-order preserving map } H : \{(i, j_{i,k}) : i, k \in \mathbb{N}\} \to \{(i, j_{i,k}) : i, k \in \mathbb{N}\} \\
\text{ such that } y_{i,\ell_{i,k}} = x_{H(i,j_{i,k})} \text{ for all } i, k \in \mathbb{N}.
\end{align*}
\]

A **regular array** in a Banach space \(X\) is an array \((x_{i,j})_{i \in \mathbb{N}, i \leq j}\) which is a basic sequence
when it is ordered with the reverse lexicographic order: \(x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}, x_{1,4}, \ldots\).
For convenience, throughout this paper, we denote the index set of a regular array by \(I\), i.e.
\[ I = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}. \] The only reason that we choose to work with \( I \) rather than \( \mathbb{N}^2 \) is because \( I \) has an enumeration (given by the reverse lexicographic order) that is easy to write down.

Notice the following:

**Remark 2.1.** If \((x_{i,j})_{(i,j)\in I}\) is a regular array and \((y_{i,j})_{(i,j)\in I}\) is a subarray of \((x_{i,j})_{(i,j)\in I}\) then \((y_{i,j})_{(i,j)\in I}\) is also regular.

The proof of the following remark can be found in functional analysis text books such as [1, Theorem 1.5.2] or [13, Lemma 1.a.5]).

**Remark 2.2.** Let \(x^*_{i,j}\) be a finite basic sequence in some infinite dimensional Banach space \(X\) having basis constant \(C\). Let \((y_i)\) be a seminormalized weakly null sequence \(X\) and \(\varepsilon > 0\). Then there exists an \(n \in \mathbb{N}\) such that \((x_1, x_2, \ldots, x_N, y_n)\) is a basic sequence with constant \(C(1 + \varepsilon)\).

By repeated application of Remark 2.2 we obtain the following.

**Remark 2.3.** Let \(X\) be a Banach space and for every \(i \in \mathbb{N}\) let \((x_{i,j})_{j=i}^{\infty}\) be a seminormalized weakly null sequence in \(X\). Then there exists a subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) which is regular. Moreover, the basis constant of \((y_{i,j})_{(i,j)\in I}\) can be chosen to be arbitrarily close to 1.

For \(p \in \mathbb{N}\) any element \(\vec{a} = (a_i)_{i=1}^{p}\) of \(\mathbb{R}^p\) will be called a \(p\)-pattern and for such \(\vec{a}\) define \(|\vec{a}| := p\). Let \((x_{i,j})_{(i,j)\in I}\) be a regular array in a Banach space \(X\). Let \(f \in X^*, k \in \mathbb{N}\), \(\vec{a} = (a_i)_{i=1}^{p}\) a \(p\)-pattern and \(F = \{j_1, j_2, \ldots, j_p\} \subseteq \mathbb{N}\). We say that \(f\) has pattern \(\vec{a}\) on \((k, F)\) with respect to \((x_{i,j})_{(i,j)\in I}\) if \(f(x_{k,j}) = a_i\) for all \(i \in \{1, 2, \ldots, p\}\).

**Lemma 2.4.** Let \((x_{i,j})_{(i,j)\in I}\) be a regular array in a Banach space \(X\), \(\vec{a}\) be a \(p\)-pattern, \(\mathcal{F} \subset 2Ba(X^*)\), \(\delta > 0\) and \(i_0, j_0, k_0 \in \mathbb{N}\) with \(j_0 \geq i_0\). Then there exists a subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) such that for any \(F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots\}\), with \((i_0, j_0) <_{rt} (k_0, \min(F))\) and \(|F| = p\) we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) and \(|g(y_{i_0,j_0})| < \delta\).

Additionally, \((y_{i,j})_{(i,j)\in I}\) can be chosen to satisfy \(y_{i,j} = x_{i,j}\) for all \((i, j) <_{rt} (i_0, j_0)\).

**Proof.** First note that there exists \(m \in \mathbb{N}\) such that for all \(f \in 2Ba(X^*)\) there exists \(j' \in \{j_0, j_0 + 1, j_0 + 2, \ldots, m\}\) with \(|f(x_{i_0,j'})| < \delta\).

Otherwise assume that for all \(m \in \mathbb{N}\) there exists \(x^*_m \in 2Ba(X^*)\) with \(|x^*_m(x_{i_0,j'})| \geq \delta\) for \(j \in \{j_0, \ldots, m\}\). By passing to a subsequence and relabeling assume that \((x^*_m)\) converges weak* to some \(x^* \in 2Ba(X^*)\). Then \(|x^*(x_{i_0,j'})| \geq \delta\) for all \(j \geq j_0\), which contradicts that each row, in particular \((x_{i_0,j'})_{j=j_0}^{\infty}\), is weakly null.

Let \(N = \{m + 1, m + 2, \ldots\}\). Divide the set \([N]^p\) of all \(p\)-element subsets of \(N\) as follows:

\[ [N]^p = \bigcup_{j=j_0}^{m+1} A_j \] where for \(j \in \{j_0, j_0 + 1, j_0 + 2, \ldots, m\}\) we set

\[ A_j = \{F \in [N]^p : \text{there exists } f \in \mathcal{F} \text{ having pattern } \vec{a} \text{ on } (k_0, F) \} \]

with respect to \((x_{i,j})_{(i,j)\in I}\) and \(|f(x_{i_0,j'})| < \delta\)

and
\( A_{m+1} = \{ F \in [N]^p : \text{there is no } f \in \mathcal{F} \text{ having pattern } \vec{a} \text{ on } (k_0, F) \text{ with respect to } (x_{i,j})_{(i,j) \in I} \}. \)

By Ramsey’s theorem there exist a subsequence \( (m_i)_{i=1}^{\infty} \subseteq [N] \), and \( j' \in \{ j_0, j_0 + 1, \ldots, m + 1 \} \) such that \( \{(m_j)_{j=1}^{\infty}\}^p \subseteq A_{j'} \) (where for an infinite subset \( M \) of \( \mathbb{N} \), \([M]\) denotes the set of all infinite subsequences of \( M \)). We then can pass to a subarray \((y_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\) by setting

\[
y_{i,j} = \begin{cases} x_{i,j} & \text{if } (i, j) <_{rt} (i_0, j_0) \\ x_{i,j'} & \text{if } (i, j) = (i_0, j_0) \\ x_{i,m_j} & \text{if } (i, j_0) <_{rt} (i, j). \end{cases}
\]

Then \((y_{i,j})_{(i,j) \in I}\) satisfies the conclusion of the lemma, since if for some \( F \subseteq \{ k_0, k_0 + 1, \ldots \} \) with \((i_0, j_0) <_{rt} (k_0, \min(F)) \) and \(|F| = p \) there exists \( f \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\), then the integer \( j' \) that was obtained by Ramsey’s theorem could not be equal to \( m + 1 \), hence \( j' \in \{ j_0, j_0 + 1, \ldots, m \} \) and the definition of \( A_{j'} \) gives the conclusion.

**Lemma 2.5.** Let \((x_{i,j})_{(i,j) \in I}\) be a regular array in a Banach space \( X \), \( \vec{a} \) be a finite set of patterns, \( \mathcal{F} \subseteq 2Ba(X^n) \), \( \delta > 0 \) and \( i_0, k_0 \in \mathbb{N} \). Then there exists some subarray \((y_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\) such that for any \( \vec{a} \in \vec{A} \), \( F \subseteq \{ k_0, k_0 + 1, k_0 + 2, \ldots \} \), with \(|F| = |\vec{a}| \) and \( j_0 \in \mathbb{N} \) with \( j_0 \geq i_0 \) and \((i_0, j_0) <_{rt} (k_0, \min(F)) \), we have the following:

If there exists \( f \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) then there exists \( g \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) and \(|g(y_{i_0,j_0})| < \delta \).

Additionally, we can assume that \( x_{i,j} = y_{i,j} \) for all \((i, j) <_{rt} (i_0, i_0)\).

**Proof.** We begin by fixing one particular element \( \vec{a} \) in \( \vec{A} \). Now apply Lemma 2.4 for \((x_{i,j})_{(i,j) \in I}, \vec{a}, \mathcal{F}, \delta, i_0, k_0 \) and \( j_0 = i_0 \) to obtain some subarray \((y_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\) with the property that for any \( F \subseteq \{ k_0, k_0 + 1, k_0 + 2, \ldots \} \), with \((i_0, j_0) <_{rt} (k_0, \min(F)) \) and \(|F| = |\vec{a}| \) we have the following. If there exists \( f \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) then there exists \( g \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) and \(|g(y_{i_0,j_0})| < \delta \). Moreover, \( y_{i,j} = x_{i,j} \) for all \((i, j) <_{rt} (i_0, j_0)\).

We repeat inductively on \( j_0 \) counting upward from \( i_0 \). Thus we next apply Lemma 2.4 to \((y_{i,j})_{(i,j) \in I}, \vec{a}, \mathcal{F}, \delta, i_0, k_0, j_0 = i_0 + 1 \) to obtain some subarray \((y_{i,j})_{(i,j) \in I}\) of \((y_{i,j})_{(i,j) \in I}\) with the property that for any \( F \subseteq \{ k_0, k_0 + 1, k_0 + 2, \ldots \} \) with \((i_0, j_0) <_{rt} (k_0, \min(F)) \) and \(|F| = |\vec{a}| \) we have the following. If there exists \( f \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) then

- there exists \( g \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) and \(|g(y_{i_0,j_0})| < \delta \) (since \( y_{i_0,j_0} = y_{i_0,i_0} \)) and
- there exists \( h \in \mathcal{F} \) having pattern \( \vec{a} \) on \((k_0, F) \) with respect to \((y_{i,j})_{(i,j) \in I}\) and \(|h(y_{i_0,j_0})| < \delta \).

Moreover, \( y_{i,j} = y_{i,j} \) for all \((i, j) <_{rt} (i_0, i_0 + 1)\). Continue in this manner for each \( j_0 \in \{ i_0 + 2, i_0 + 3, \ldots \} \). Note that by fixing the elements of the subarray for \((i, j) <_{rt} (i_0, j_0)\) at each step \( j_0 \), there exists a subarray after infinitely
many steps which possesses the properties of all the previous subarrays. We call this “limit” subarray \((y_{i,j}^\ddagger)_{(i,j)\in I}\) and notice it has the property that for any \(F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots\}\) with \(|F| = |\vec{a}|\) and for all \(j_0 \in \mathbb{N}\) with \((i_0, j_0) <_\ell (k_0, \min(F))\), we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^\ddagger)_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^\ddagger)_{(i,j)\in I}\) and \(|g(y_{i_0,j_0}^\ddagger)| < \delta\).

Notice also that any further subarray of \((y_{i,j}^\ddagger)_{(i,j)\in I}\) has this same property. Then repeat the above process for each \(\vec{a} \in \vec{A}\) to obtain the desired array. \(\Box\)

Notice that if \((y_{i,j})_{(i,j)\in I}\) is the result of applying Lemma 2.4 to some regular array and \((z_{i,j})_{(i,j)\in I}\) is a subarray of \((y_{i,j})_{(i,j)\in I}\) then \((z_{i,j})_{(i,j)\in I}\) does not necessarily retain the property in the conclusion of Lemma 2.4. However, if \((y_{i,j})_{(i,j)\in I}\) is the result of applying Lemma 2.5 to some regular array and \((z_{i,j})_{(i,j)\in I}\) is a regular subarray of \((y_{i,j})_{(i,j)\in I}\) then \((z_{i,j})_{(i,j)\in I}\) does retain the property in the conclusion of Lemma 2.5. This idea is summarized in the following remark.

Remark 2.6. Let \((x_{i,j})_{(i,j)\in I}\) be a regular array in a Banach space \(X\), \(\vec{A}\) be a finite set of patterns, \(\mathcal{F} \subseteq 2\mathcal{B}(X^*)\), \(i_0, k_0 \in \mathbb{N}\) and \(\delta > 0\). Then there exists a subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) such that if \((z_{i,j})_{(i,j)\in I}\) is any subarray of \((y_{i,j})_{(i,j)\in I}\), then for any \(\vec{a} \in \vec{A}\), \(F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots\}\), with \(|F| = |\vec{a}|\) and \(j_0 \in \mathbb{N}\) with \((i_0, j_0) <_\ell (k_0, \min(F))\), we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((z_{i,j})_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((z_{i,j})_{(i,j)\in I}\) and \(|g(z_{i_0,j_0})| < \delta\).

Additionally, we can assume that \(x_{i,j} = y_{i,j}\) for all \((i, j) <_\ell (i_0, i_0)\).

Lemma 2.7. Let \((x_{i,j})_{(i,j)\in I}\) be a regular array in a Banach space \(X\), \(\vec{A}\) be a finite set of patterns, \(\mathcal{F} \subseteq 2\mathcal{B}(X^*)\) and \(\delta > 0\). Then there exists some subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) such that for all \(\vec{a} \in \vec{A}\), \(k_0 \in \mathbb{N}\), \(F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots\}\) with \(|F| = |\vec{a}|\) and \((i_0, j_0) \in I\) with \((1, k_0) \leq \ell (i_0, j_0) <\ell (k_0, \min(F))\) we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) and \(|g(y_{i_0,j_0})| < \delta\).

Proof. We will apply Remark 2.6 inductively with the subarray changing at each step, but \(\vec{A}\), \(\mathcal{F}\) and \(\delta\) remaining as in the hypothesis and \((i_0, k_0)\) cycling through \(\mathbb{N}^2\). We create the final subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) inductively one column at a time. At the \(j_0\) step of the induction we create a subarray \((y_{i,j})_{(i,j)\in I}\) of \((y_{i,j}^{j_0})_{(i,j)\in I}\) (where for \(j_0 = 1, (y_{i,j}^{j_0})_{(i,j)\in I} = (x_{i,j})_{(i,j)\in I}\) and we set \(y_{i,j_0} = y_{i,j_0}^{j_0}\) for \(i \in \{1, 2, \ldots, j_0\}\).

COLUMN 1: Apply Remark 2.6 to \((x_{i,j})_{(i,j)\in I}, \vec{A}\), \(\mathcal{F}\), \(\delta\), \(i_0 = 1\), and \(k_0 = 1\) to obtain a subarray \((y_{i,j})_{(i,j)\in I}\) with the property that for all \(\vec{a} \in \vec{A}\), \(F \subseteq \{1, 2, \ldots\}\) with \(|F| = |\vec{a}|\) such that \((1, 1) <\ell (k_0, \min(F))\) we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) and \(|g(y_{1,1})| < \delta\).
We then fix column 1 of \((y_{i,j})_{(i,j)\in I}\) by setting \(y_{1,1} := y_{1,1}^1\).

**COLUMN 2:** Apply Remark 2.6 to \((y_{i,j+1}^1)_{(i,j)\in I}, \vec{A}, \mathcal{F}, \delta\), successively for each \((i_0, k_0) \in \{(1,2), (2,1), (2,2)\}\) to obtain a subarray \((y_{i,j}^2)_{(i,j)\in I}\) with the property that for all \(\vec{a}\) in \(\vec{A}\), \(F \subseteq \{2,3,\ldots\}\) with \(|F| = |\vec{a}|\), \(i_0 \in \{1,2\}\) and \(k_0 \in \{1,2\}\) such that \((i_0,2) <_{rt} (k_0, \text{min}(F))\) we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^2)_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^2)_{(i,j)\in I}\) and \(|g(y_{i,j}^2_{1,2})| < \delta\).

We then fix column 2 of \((y_{i,j})_{(i,j)\in I}\) by setting \(y_{i,2} := y_{i,2}^2\) for \(i \in \{1,2\}\).

**COLUMN \(j_0\):** Apply Remark 2.6 to \((y_{i,j+1}^{j_0-1})_{(i,j)\in I}, \vec{A}, \mathcal{F}, \delta\), successively for each \((i_0, k_0) \in \{(i_0,j_0) : 1 \leq i < j_0\} \cup \{(j_0,j) : 1 \leq j \leq j_0\}\) to obtain a subarray \((y_{i,j}^{j_0})_{(i,j)\in I}\) with the property that for all \(\vec{a}\) in \(\vec{A}\), \(F \subseteq \{j_0,j_0+1,\ldots\}\) with \(|F| = |\vec{a}|\), \(i_0 \in \{1,2,\ldots,j_0\}\) and \(k_0 \in \{1,2,\ldots,j_0\}\) such that \((i_0, j_0) <_{rt} (k_0, \text{min}(F))\) we have the following:

If there exists \(f \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^{j_0})_{(i,j)\in I}\) then there exists \(g \in \mathcal{F}\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j}^{j_0})_{(i,j)\in I}\) and \(|g(y_{i,j}^{j_0}_{1,2})| < \delta\).

We then fix column \(j_0\) of \((y_{i,j})_{(i,j)\in I}\) by setting \(y_{i,j_0} := y_{i,j_0}^{j_0}\) for \(i \in \{1,2,\ldots,j_0\}\).

Let \(\vec{a}\) in \(\vec{A}\), \(k_0 \in \mathbb{N}\), \(F \subseteq \{k_0,k_0 + 1, k_0 + 2, \ldots\}\) with \(|F| = |\vec{a}|\) and \((i_0,j_0) \in I\) with \((1,k_0) \leq_{rt} (i_0,j_0) <_{rt} (k_0, \text{min}(F))\) all be given. Since \((1,k_0) \leq_{rt} (i_0,j_0)\) we have that \(k_0 \leq j_0\). Since \((i_0,j_0) <_{rt} (k_0, \text{min}(F))\) we have that there exists a set \(G \subseteq \mathbb{N}\) with \(|G| = |F|\), \(\text{min}(G) \geq k_0\) and \((y_{k_0,j})_{j\in F} = (y_{k_0,j}^{j_0})_{j\in G}\). Thus if there exists \(f \in \mathcal{F}\) which has pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) then \(f\) has pattern \(\vec{a}\) on \((k_0, G)\) with respect to \((y_{i,j}^{j_0})_{(i,j)\in I}\), therefore by the property of \((y_{i,j}^{j_0})_{(i,j)\in I}\) we obtain that there exists \(g \in \mathcal{F}\) which has pattern \(\vec{a}\) on \((k_0, G)\) with respect to \((y_{i,j}^{j_0})_{(i,j)\in I}\) and \(|g(y_{i,j}^{j_0}_{1,2})| < \delta\). Hence \(g\) has pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) and \(|g(y_{i,j})| < \delta\).

**Lemma 2.8.** Let \((x_{i,j})_{(i,j)\in I}\) be a regular array in a Banach space \(X\), \(\varepsilon > 0\), and \(k \in \mathbb{N}\). Then there exists some subarray \((y_{i,j})_{(i,j)\in I}\) of \((x_{i,j})_{(i,j)\in I}\) such that for any pattern \(\vec{a}\) in \([-1,1]^p\) for some \(p \leq k\), for any \(k_0 \in \mathbb{N}\) and any \(F \subseteq \{k_0,k_0 + 1, k_0 + 2, \ldots\}\) with \(|F| = |\vec{a}|\) we have the following:

If there exists \(f \in \text{Ba}X^*\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) then there exists \(g \in (1+\varepsilon)\text{Ba}X^*\) having pattern \(\vec{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j)\in I}\) and

\[
\sum_{\{(i,j)\in I : j \geq k_0\} \setminus \{(k_0,\ell) : \ell\in F\}} |g(y_{i,j})| < \varepsilon
\]

**Proof.** Let \((\delta_j)_{j=0}^\infty \subseteq (0,1)\) such that

\[
\frac{1}{\inf_{i,j} \|x_{i,j}\|} \left(4Ck\delta_0 + \sum_{j=1}^\infty 4Cj\delta_j\right) < \varepsilon
\]

(9)
where \( C \) is the basis constant for the regular array \((x_{i,j})_{(i,j) \in I}\). Let \( A_0 \) be a \( \delta_0 \) net for \([-1,1] \)
containing zero and for each \( j \in \mathbb{N} \) choose a \( \delta_j \) net \( B_j \) for \([-1,1] \) with \( \{0\} \subseteq A_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \). Let

\[
(10) \quad \vec{A} = \{ \vec{a} = (a_i)_{i=1}^p \in A_0^p : \text{where } 1 \leq p \leq k \}
\]

and

\[
(11) \quad \mathcal{F} = \{ f \in (1 + \frac{\varepsilon}{2}) \mathcal{B}(X^*) : f(x_{i,j}) \in B_j \text{ for all } (i,j) \in I \},
\]

where with out loss of generality we assume \( \varepsilon < 2 \) so \( \mathcal{F} \subseteq 2 \mathcal{B}(X^*) \). Since \( 0 \in A_0 \) the zero functional is in \( \mathcal{F} \) therefore \( \mathcal{F} \) is nonempty.

We construct the subarray \((y_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\) inductively. First we will construct a subarray \((y^1_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\), then for \( j \in \mathbb{N} \) for \( j \geq 2 \) we will construct a subarray \((y^j_{i,k})_{(i,k) \in I}\) of \((y^{j-1}_{i,k+j-1})_{(i,k) \in I}\). Once the subarray \((y^j_{i,k})_{(i,k) \in I}\) has been constructed we set \( y_{i,j} := y^j_{i,j} \) for \( 1 \leq i \leq j \). Since \( y^1_{i,k} \) is a subarray of \((y^{j-1}_{i,k+j-1})_{(i,k) \in I}\) and \( y_{i,j} = y^j_{i,j} \) for \( 1 \leq i \leq j \), we have that \((y_{i,j})_{(i,j) \in I}\) is a subarray of \((x_{i,j})_{(i,j) \in I}\).

Apply Lemma 2.7 to \((x_{i,j})_{(i,j) \in I}\), \( \vec{A} \), \( \delta_1 \), \( \mathcal{F} \) to obtain a subarray \((y^1_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\) such that for all \( \vec{a} \in \vec{A} \), \( k_0 \in \mathbb{N} \), \( F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots \} \) with \( |F| = |\vec{a}| \) and \((i_0,j_0) \in I \)
with \( (1, k_0) \leq_{rt} (i_0, j_0) <_{rt} (k_0, \min(F)) \) we have: if there exists \( f \in \mathcal{F} \) having pattern \( \vec{a} \in \vec{A} \)
on \((k_0,F)\) with respect to \((y^1_{i,j})_{(i,j) \in I}\) then there exists \( g \in \mathcal{F} \) having pattern \( \vec{d} \in \vec{A} \) on \((k_0,F)\)
with respect to \((y^1_{i,j})_{(i,j) \in I}\) and \( |g(y^1_{i_0,j_0})| < \delta_1 \). Define the elements of the first column of \((y_{i,j})_{(i,j) \in I}\) by setting \( y_{1,1} := y^1_{1,1} \). Define for each \( b \in B_1 \) the set

\[
(12) \quad \mathcal{F}_b = \{ f \in \mathcal{F} : f(y_{1,1}) = b \}.
\]

Apply Lemma 2.7 to \((y^1_{i,j+1})_{(i,j) \in I}\), \( \vec{A} \), \( \delta_2 \), \( \mathcal{F}_b \) successively for each \( b \in B_1 \) to obtain a subarray \((y^2_{i,j})_{(i,j) \in I}\) of \((y^1_{i,j+1})_{(i,j) \in I}\) such that for all \( \vec{a} \in \vec{A} \), \( k_0 \in \mathbb{N} \), \( F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots \} \) with \( |F| = |\vec{a}| \) and \((i_0,j_0) \in I \)
with \( (1, k_0) \leq_{rt} (i_0, j_0) <_{rt} (k_0, \min(F)) \) we have: if there exists \( f \in \mathcal{F}_b \) having pattern \( \vec{a} \in \vec{A} \) on \((k_0,F)\)
with respect to \((y^1_{i,j+1})_{(i,j) \in I}\) then there exists \( g \in \mathcal{F}_b \) having pattern \( \vec{d} \in \vec{A} \) on \((k_0,F)\)
with respect to \((y^2_{i,j})_{(i,j) \in I}\) and \( |g(y^1_{i_0,j_0})| < \delta_2 \). Define the elements of the second column of \((y_{i,j})_{(i,j) \in I}\) by setting \( y_{1,2} := y^2_{1,2} \), and \( y_{2,2} := y^2_{2,2} \). For each \( \vec{b} = (b_1, b_2, b_3) \in B_1 \times B_2 \times B_2 \) set

\[
(13) \quad \mathcal{F}_b = \{ f \in \mathcal{F} : f(y_{1,1}) = b_1, f(y_{1,2}) = b_2 \text{ and } f(y_{2,2}) = b_3 \}.
\]

Apply Lemma 2.7 to \((y^2_{i,j+2})_{(i,j) \in I}\), \( \vec{A} \), \( \delta_3 \), \( \mathcal{F}_b \) successively for each \( \vec{b} \in B_1 \times B_2 \times B_2 \) to obtain a subarray \((y^3_{i,j})_{(i,j) \in I}\) of \((y^2_{i,j+2})_{(i,j) \in I}\) such that for all \( \vec{a} \in \vec{A} \), \( k_0 \in \mathbb{N} \), \( F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots \} \) with \( |F| = |\vec{a}| \) and \((i_0,j_0) \in I \)
with \( (1, k_0) \leq_{rt} (i_0, j_0) <_{rt} (k_0, \min(F)) \) we have: if there exists \( f \in \mathcal{F}_b \) having pattern \( \vec{a} \in \vec{A} \) on \((k_0,F)\)
with respect to \((y^2_{i,j+2})_{(i,j) \in I}\) then there exists \( g \in \mathcal{F}_b \) having pattern \( \vec{d} \in \vec{A} \) on \((k_0,F)\)
with respect to \((y^3_{i,j})_{(i,j) \in I}\) and \( |g(y^2_{i_0,j_0})| < \delta_3 \). Define the elements in the third column \((y_{i,j})_{(i,j) \in I}\)
by setting \( y_{1,3} := y^3_{1,3} \), \( y_{2,3} := y^3_{2,3} \) and \( y_{3,3} := y^3_{2,3} \). Continue in this manner to create the subarray \((y_{i,j})_{(i,j) \in I}\) of \((x_{i,j})_{(i,j) \in I}\).

Let \( \vec{f} \in \mathcal{B}(X^*) \), \( \vec{c} \) be a \( p \)-pattern for \( p \leq k \), \( k_0 \in \mathbb{N} \) and \( F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots \} \) with \( |F| = p \) such that \( \vec{f} \) has pattern \( \vec{c} \) on \((k_0,F)\) with respect to \((y_{i,j})_{(i,j) \in I}\).

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First it is easy to see using (9) that since $\tilde{f} \in Ba(X^*)$ there is $f \in F$ (as defined in (11)) such that

- for all $j \in F$ we have $f(y_{k,0,j}) \in A_0$ and $|\tilde{f}(y_{k,0,j}) - f(y_{k,0,j})| \leq \delta_0$, and
- if we define the finite set $G \subseteq \mathbb{N}$ by $(y_{k,0,j})_{j \in F} = (x_{k,0,j})_{j \in G}$, then for all $(i, j) \in I \setminus \{(k_0, j) : j \in G\}$ we have that $|\tilde{f}(x_{i,j}) - f(x_{i,j})| \leq \delta_j$ and $f(x_{i,j}) \in B_j$.

Let $\vec{a} := (f(y_{k,0,j}))_{j \in F}$ and note that $f$ has pattern $\vec{a}$ on $(k_0, F)$ with respect to $(y_{i,j})_{(i,j) \in I}$. We will find a functional $g \in (1 + \frac{\varepsilon}{2})Ba(X^*)$ such that $g$ has pattern $\vec{a}$ on $F$ with respect to $(y_{i,j})_{(i,j) \in I}$ and $\sum_{(k_0,j) \in F} |g(y_{i,j})| < \varepsilon$.

We proceed to find such functional $g$. But first a bit of notation, for a $p$-pattern $\vec{a} = (a_i)_{i=1}^p$ we define its derivative $\vec{a}' = (a_{i-1})_{i=2}^p$.

We will walk through the index set $I' = \{(i, j) \in I : j \geq k_0\}$ proceeding through this set in $\prec_F$-order and at each step find a functional $g_{i,j}$ with the property that if $(i, j) \notin \{(k_0, j) : j \in F\}$ then $g_{i,j}(y_{i,j})$ will be small and “agree” with the previous functional on $\{(i', j') \in I' : (i', j') \prec_F (i, j)\}$. If $(i, j) \in \{(k_0, j) : j \in F\}$ then we will not change the previously defined functional. We will assume $k_0 \geq 3$ for purposes of demonstrating the construction, but if $k_0 = 1$ or 2 then we proceed similarly.

**STEP (1, $k_0$):** Note $\{(k_0, j) : j \in F\}$ (since $k_0 \geq 3$). Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), f(y_{3,1}), \ldots, f(y_{k_0-1,k_0-1})).$$

Then $f \in F_{\vec{b}}$, $f$ has pattern $\vec{a}$ on $(k_0, F)$ with respect to $(y_{i,j})_{(i,j) \in I}$, $(y_{i,j})_{(i,j) \in I'}$ is a subarray of $(y_{i,j})_{(i,j) \in I}$ and $(1, k_0) \prec_F (1, k_0, \min(F))$, (the last inequality is valid since $k_0 \geq 3$). Thus there exists $g_{1,k_0} \in F_{\vec{b}}$ such that $g_{1,k_0}$ has pattern $\vec{a}$ on $(k_0, F)$ with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{1,k_0}(y_{i,j})| < \delta_{k_0}$. Set $F_{1,k_0} = F$ and $\vec{a}_{1,k_0} = \vec{a}$.

**STEP (2, $k_0$):** Note that $(2, k_0) \notin \{(k_0, j) : j \in F_{1,k_0}\}$ (since $k_0 \geq 3$). Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), \ldots, f(y_{k_0-1,k_0-1}), g_{1,k_0}(y_{1,k_0})).$$

Then $g_{1,k_0} \in F_{\vec{b}_{\text{verb}}}$, $g_{1,k_0}$ has pattern $\vec{a}_{1,k_0}$ on $(k_0, F_{1,k_0})$ with respect to $(y_{i,j})_{(i,j) \in I}$ and $(1, k_0) \prec_F (2, k_0, \min(F))$, (the last inequality is valid because $k_0 \geq 3$). Thus there exists $g_{2,k_0} \in F_{\vec{b}}$ such that $g_{2,k_0}$ has pattern $\vec{a}_{1,k_0}$ on $(k_0, F_{1,k_0})$ with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{2,k_0}(y_{2,k_0})| < \delta_{k_0}$. Set $F_{2,k_0} = F_{1,k_0}$ and $\vec{a}_{2,k_0} = \vec{a}_{1,k_0}$.

We continue similarly until the $(k_0 - 1, k_0)$ step. The step $(k_0, k_0)$ is slightly different. We separate this step into two different cases depending on whether or not $(k_0, k_0) \in \{(k_0, j) : j \in F_{k_0-1,k_0}\}$.

**STEP ($k_0, k_0$):** If $(k_0, k_0) \in \{(k_0, j) : j \in F_{k_0-1,k_0}\}$ then set $g_{k_0,k_0} = g_{k_0-1,k_0}$, $F_{k_0,k_0} = F_{k_0-1,k_0} \setminus \{k_0\}$ and $\vec{a}_{k_0,k_0} = \vec{a}_{k_0-1,k_0}$.

If $(k_0, k_0) \notin \{(k_0, j) : j \in F_{k_0-1,k_0}\}$ then $(k_0, k_0) \prec_F (k_0, \min(F))$. Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), \ldots, f(y_{k_0-1,k_0-1}), g_{1,k_0}(y_{1,k_0}), g_{2,k_0}(y_{2,k_0}), \ldots, g_{k_0-1,k_0}(y_{k_0-1,k_0})).$$

Then $g_{k_0-1,k_0} \in F_{\vec{b}}$, $g_{k_0-1,k_0}$ has pattern $\vec{a}_{k_0-1,k_0}$ on $(k_0, F_{k_0-1,k_0})$ with respect to $(y_{i,j})_{(i,j) \in I}$, $(y_{i,j})_{(i,j) \in I'}$ is a subarray of $(y_{i,j})_{(i,j) \in I}$ and $(1, k_0) \leq_F (k_0, k_0, \min(F))$. Thus there exists $g_{k_0,k_0} \in F_{\vec{b}}$ such that $g_{k_0,k_0}$ has pattern $\vec{a}_{k_0-1,k_0}$ on $(k_0, F_{k_0-1,k_0})$ with respect to $(y_{i,j})_{(i,j) \in I}$ and $g_{k_0,k_0}(y_{k_0,k_0}) < \delta_{k_0}$. In this case set $F_{k_0,k_0} = F_{k_0-1,k_0}$ and $\vec{a}_{k_0,k_0} = \vec{a}_{k_0-1,k_0}$.

Then start again with the first entry $(1, k_0 + 1)$ of the next column as in steps (1, $k_0$) and (2, $k_0$).
Proof. Since \( g_{k_0, k_0} \in F_{k_0, k_0} \) has pattern \( \tilde{a}_{k_0, k_0} \) on \( (k_0, F_{k_0, k_0}) \) with respect to \( (y_{i,j})(i,j) \in I \), \( (y_{i,j})(i,j) \in I' \) is a subarray of \( (g_{k_0})_{i,j} \in I \) and \( (1, k_0) \leq \rho \ell (1, k_0 + 1) < \rho \ell (k_0, \min(F)) \), the last inequality is valid since \( k_0 \geq 3 \). Thus there exists \( g_{1, k_0 + 1} \in F_{k_0, k_0} \) such that \( g_{1, k_0 + 1} \) has pattern \( \tilde{a}_{k_0, k_0} \) on \( (k_0, F_{k_0, k_0}) \) with respect to \( (y_{i,j})(i,j) \in I \) and \( |g_{1, k_0 + 1}(y_{i,j})| < \delta_{k_0 + 1} \). Set \( F_{1, k_0 + 1} = F_{k_0, k_0} \) and \( \tilde{a}_{1, k_0 + 1} = \tilde{a}_{k_0, k_0} \).

Continue in this manner to generate a sequence of functionals \( (g_{i,j})(i,j) \in I' \). We only need to distinguish two cases every time we reach the \( k_0 \) row as in step \( (k_0, k_0) \). Let \( g \in (1 + \frac{\varepsilon}{\ell})Ba(X^*) \) be a weak*-accumulation point of sequence \( (g_{i,j})(i,j) \in I' \). Note \( g \) has the following two properties:

1. \( g \) has pattern \( \tilde{a} \) on \( (k_0, F) \) with respect to \( (y_{i,j})(i,j) \in I \), and \( \sum_{(i,j) \in I \setminus \{(k_0,\ell), \ell \in F\}} |g(y_{i,j})| < \varepsilon \).

Since \( g(y_{k_0,\ell}) = f(y_{k_0,\ell}) \in A_0 \) for all \( \ell \in F \) and \( |f(y_{k_0,\ell}) - \tilde{f}(y_{k_0,\ell})| < \delta_0 \) for all \( \ell \in F \), (9) implies that there exists \( \tilde{g} \in X^* \) such that \( \|\tilde{g}\| \leq \|g\| + \frac{\varepsilon}{2} \leq 1 + \varepsilon \) and \( \tilde{g}(y_{k_0,\ell}) = \tilde{f}(y_{k_0,\ell}) \in A_0 \) for all \( \ell \in F \) (thus \( \tilde{g} \) has pattern \( \tilde{c} \) on \( (k_0, F) \) with respect to \( (y_{i,j})(i,j) \in I \)) and

\[
\sum_{(i,j) \in I \setminus \{(k_0,\ell), \ell \in F\}} |\tilde{g}(y_{i,j})| < \varepsilon
\]

completing the proof. \( \square \)

**Theorem 2.9.** Let \( (x_{i,j})(i,j) \in I \) be a regular array in a Banach space \( X \), \( (M_j)_{j \in \mathbb{N}} \subseteq \mathbb{N} \) be an increasing sequence of integers and \( \varepsilon > 0 \). Then there exists a regular subarray \( (y_{i,j})(i,j) \in I \) of \( (x_{i,j})(i,j) \in I \) such that for any finitely supported scalars \( (a_{i,j})(i,j) \in I \), \( k_0 \in \mathbb{N} \) and \( F \subseteq \mathbb{N} \) with \( |F| \leq M_{\min(F)} \) and \( k_0 \leq \min(F) \) we have

\[
\| \sum_{(i,j) \in I} a_{i,j}y_{i,j} \| \geq \frac{1}{2 + \varepsilon} \| \sum_{j \in F} a_{k_0,j}y_{k_0,j} \|
\]

**Proof.** Let \( \eta > 0 \) such that

\[
2(2\eta + 1) \leq 2 + \frac{\varepsilon}{2}
\]

where \( C \) is the basis constant of the regular array \( (x_{i,j})(i,j) \in I \). Apply Lemma 2.8 to \( (x_{i,j})(i,j) \in I \), \( \eta \) and \( M_1 \) to get \( (y_{i,j}^1)(i,j) \in I \). Define \( y_{1,1} := y_{1,1}^1 \).

Apply Lemma 2.8 to \( (y_{i,j}^1)(i,j) \in I \), \( \eta \) and \( M_2 \) to get \( (y_{i,j}^2)(i,j) \in I \). Define \( y_{i,2} := y_{i,2}^2 \) for \( i = 1, 2 \).

Assuming that \( (y_{i,j}^{\ell-1})(i,j) \in I \) has been defined (and thus \( (y_{i,j})(i,j) \in I \) has also been defined) apply Lemma 2.8 to \( (y_{i,j}^{\ell-1})(i,j) \in I \), \( \eta \) and \( M_\ell \) to get \( (y_{i,j}^\ell)(i,j) \in I \). Define \( y_{i,\ell} := y_{i,\ell}^\ell \) for \( i = 1, 2, \ldots, \ell \).

Inductively construct the entire array \( (y_{i,j})(i,j) \in I \) and notice \( (y_{i,j})(i,j) \in I \) is regular by Remark 2.1.

Let \( k_0 \in \mathbb{N}, F \subseteq \{k_0, k_0 + 1, k_0 + 2, \ldots\} \) with \( |F| \leq M_{\min(F)} \) and finitely supported scalars \( (a_{i,j})(i,j) \in I \) be given. We can assume without loss of generality that
\[ \| \sum_{(i,j) \in I} a_{i,j} y_{i,j} \| = 1. \]

Then \(|a_{i,j}| \leq 2C\) for \((i, j) \in I\). Let \(f \in Ba(X^*)\) such that

\[ f \left( \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right) = \| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \|. \]

Let \(\tilde{a} = (f(y_{k_0,j}))_{j \in F}\) be a \(p\)-pattern where \(p = |F|\). Obviously \(f\) has pattern \(\tilde{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j) \in I}\). Then by considering the subarray \((y_{i,j})_{(i,j) \in I, j \geq \min(F)}\) of \((y_{i,j}^{\min(F)})_{(i,j) \in I, j \geq \min(F)}\) we obtain by the above that there exists \(g \in (1+\eta)Ba(X^*)\) having pattern \(\tilde{a}\) on \((k_0, F)\) with respect to \((y_{i,j})_{(i,j) \in I}\) and \(\sum\{(i,j) \in I, j \geq \min(F)\} \setminus \{(k_0,j) : j \in F\} \ |g(y_{i,j})| < \eta\). Thus

\[
1 = \| \sum_{j \geq \min(F)} a_{i,j} y_{i,j} \| \geq \frac{1}{2C(1+\eta)} \left( \sum_{j \in F} a_{k_0,j} g(y_{k_0,j}) \right) = \frac{1}{2C(1+\eta)} \left( \sum_{j \in F} a_{i,j} g(y_{i,j}) \right)
\]

\[
\geq \frac{1}{2C(1+\eta)} \left( \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right) - \frac{1}{2C(1+\eta)} \sum_{\{(i,j) \in I, j \geq \min(F)\} \setminus \{(k_0,j) : j \in F\}} |a_{i,j}| |g(y_{i,j})|\]

\[
\geq \frac{1}{2C(1+\eta)} \left( \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right) - \frac{1}{2C(1+\eta)} \sum_{\{(i,j) : j \geq \min(F)\} \setminus \{(k_0,j) : j \in F\}} |g(y_{i,j})|\]

Thus by (14) and (15) we have

\[
\| \sum_{(i,j) \in I} a_{i,j} y_{i,j} \| = 1 \geq \frac{1}{2C(2\eta + 1)} \left( \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right) \geq \frac{1}{C(2 + \frac{\epsilon}{2})} \left( \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right).
\]

Since we can choose \(C\), the basis constant of our regular array, arbitrarily close to 1 (see Remark 2.3) we have shown the result.

\[ \square \]

3. Existence of Non-trivial Operators

In this section we will prove Theorem 0.7 which is one of the main results of the paper. Theorem 2.9 will play an important role in its proof (see the proof of Lemma 3.3).

For the proof of Theorem 0.7 we will need the following result which also gives sufficient conditions for a Banach space \(X\) so that the “multiple of the inclusion plus compact” problem to have an affirmative answer in \(X\).

**Theorem 3.1.** Let \(X\) be a Banach space containing seminormalized basic sequences \((x_i)\) and \((x^*_i)\) for all \(n \in \mathbb{N}\), such that \(0 < \inf_{n,i} \|x^*_i\| \leq \sup_{n,i} \|x^*_i\| < \infty\). Let \((x_i)\) and \((z_i)\) be seminormalized basic sequences not necessarily in \(X\). Assume the following:
• The sequence \((x_i)\) is dominated by the sequence \((x_i)\).
• The sequence \((x_i)\) satisfies condition (5) of Theorem 0.7 and has Property P2.
• For all \(n \in \mathbb{N}\) the sequence \((x^n_i)\), satisfies condition (6) of Theorem 0.7.

Then there exists a subspace \(Y\) of \(X\) which has a basis and an operator \(T \in \mathcal{L}(Y, X)\) which is not a compact perturbation of a multiple of the inclusion map.

In order to prove Theorem 3.1 we need the following two lemmas whose proofs are postponed.

**Lemma 3.2.** Let \(X\) be a Banach space, \((x_n)\) be a seminormalized basic sequence in \(X\) having Property P2 and \((z_n)\) be a seminormalized basic sequence not necessarily in \(X\). Assume that the sequence \((x_n)\) satisfies condition (5) of Theorem 0.7. Then for all \((\delta_n)_{n=2}^{\infty} \subseteq (0, \infty)\) there exists an increasing sequence \(M_1 < M_2 < \cdots\) of positive integers and a subsequence \((x_{n_i})\) of \((x_i)\) such that for all \((a_i) \in c_00\),

\[
\| \sum a_i x_{n_i} \| \leq \sup_{n \in \mathbb{N}} \sup_{n \leq F \subseteq N \mid F \leq M_n} \delta_n \| \sum_{i \in F} a_i z_i \|,
\]

for some \(\delta_1\) (where \(\delta n \leq F\)” means \(n \leq \min(F)\)).

**Lemma 3.3.** Let \(X\) be a Banach space, \((\delta_n)\) be a summable sequence of positive numbers, \((M_n)_{n=1}^{\infty} \subseteq \mathbb{N}\) be a sequence of positive integers, \((z_n)\) be a seminormalized basic sequence (not necessarily in \(X\)) and for every \(n \in \mathbb{N}\) let \((x^n_j)_{j=1}^{\infty}\) be a weakly null basic sequence in \(X\) having spreading model \((x^n_j)_{j=1}^{\infty}\) such that \(0 < \inf_{n,j} \| x^n_j \| \leq \sup_{n,j} \| x^n_j \| < \infty\) and condition (6) of Theorem 0.7 is satisfied. Then there exists a seminormalized weakly null basic sequence \((y_i)\) in \(X\) such that

\[
\frac{1}{6C} \sup_{n} \sup_{n \leq F \subseteq N \mid F \leq M_n} \delta_n \| \sum_{i \in F} a_i z_i \| \leq \| \sum a_i y_i \|.
\]

Moreover, \(\| y_i \| \geq \frac{\delta_1}{2} \inf_{n,m} \| x^n_m \|\). Furthermore, if \((x^n_i)\) is any given sequence of functionals in \(X^*\) and \(\varepsilon > 0\) we can choose \((y_i)\) to satisfy \(|x^n_i y_i| < \varepsilon\).

We now present the

**Proof of Theorem 3.1.** Note that the assumptions of Lemmas 3.2 and 3.3 are almost included in the assumptions of Theorem 3.1, with the exception that in Lemma 3.3 the sequence \((x^n_i)\) is assumed to be weakly null for all \(n\). We will replace the sequences \((x_i)\), \((x^n_i)\), \((x_i)\) and \((z_i)\) by sequences \((X_i)\), \((X^n_i)\), \((X_i)\) and \((Z_i)\) respectively, such that \((X_i)\), \((X^n_i)\) and \((Z_i)\) satisfy the assumptions of Lemma 3.2 and 3.3. Notice that if \(\ell_1\) embeds in \(X\) then \(X\) contains an unconditional basic sequence thus as we mentioned in the Introduction, the conclusion of Theorem 3.1 is valid in this case. Therefore we can assume that \(\ell_1\) does not embed in \(X\). Then by Rosenthal’s \(\ell_1\) Theorem [17] and a diagonal argument, by passing to subsequences of \((x_i)\), \((x^n_i)\) and \((z_i)\) and relabeling, we can assume that \((x_i)\) and \((x^n_i)\) are weakly Cauchy for all \(n \in \mathbb{N}\). Let \((X_i) := (x_{2i} - x_{2i-1})\), \((X^n_i) := (x^n_{2i} - x^n_{2i-1})\), \((X_i) := (x_{2i} - x_{2i-1})\) and \((Z_i) := (z_{2i} - z_{2i-1})\). It is trivial to check that all the assumptions of Lemma 3.2 and 3.3 are satisfied for the sequences \((X_i)\), \((X^n_i)\) and \((Z_i)\). Thus assume that this is the case for the original sequences \((x_i)\), \((x^n_i)\) and \((z_i)\). Let \((\delta_n)_{n=2}^{\infty}\) be a summable sequence of positive numbers. First apply Lemma 3.2 to obtain a subsequences \((x_{n_i})\), \(\delta_1 > 0\) and an increasing
sequence \((M_n)_{n \in \mathbb{N}}\) of positive integers which satisfies (16). For every \(i \in \mathbb{N}\) let a norm 1 functional \(x_i^*\) satisfying \(x_i^* x_n = \|x_n\|\). Then apply Lemma 3.3 for \((\delta_n)_{n \in \mathbb{N}}\) and \((M_n)_{n \in \mathbb{N}}\) to obtain a basic sequence \((y_i)\) which satisfies (17).

Assume also that \(|y_i|\) satisfies the “furthermore” part of the statement of Lemma 3.3 for the sequence \((x_i^*)\) and \(\varepsilon = \delta_1(\inf_{n,i} \|x^n_i\|)\). Note that if \(|\lambda| \geq \frac{4 \sup_{n,i} \|x^n_i\|}{\delta_1 \inf_{n,i} \|x^n_i\|}\) then

\[
\|x_n + \lambda y_i\| \geq |\lambda| \|y_i\| - \|x_n\| \geq \frac{4 \sup_{n,i} \|x^n_i\|}{\delta_1 \inf_{n,i} \|x^n_i\|} \inf_{n,i} \|x^n_i\| - \|x_n\| \geq \|x^n_i\| \geq \inf_{n,i} \|x^n_i\|
\]

(by the “moreover” part of the statement of Lemma 3.3). Also if \(|\lambda| < \frac{4 \sup_{n,i} \|x^n_i\|}{\delta_1 \inf_{n,i} \|x^n_i\|}\) then

\[
\|x_n + \lambda y_i\| \geq |\lambda| (\|x_n\| + \|y_i\|) \geq \|x_n\| - \frac{4 \sup_{n,i} \|x^n_i\|}{\delta_1 \inf_{n,i} \|x^n_i\|} \inf_{n,i} \|x^n_i\| \varepsilon \geq \frac{1}{2} \inf_{n,i} \|x^n_i\|.
\]

Thus for all scalars \(\lambda\) we have

\[
\|x_n + \lambda y_i\| \geq \frac{1}{2} \inf_{n,i} \|x^n_i\|.
\]

Thus if we define \(T : [(y_i)] \to X\) by

\[T(\sum a_i y_i) = \sum a_i x_n\]

we have that this operator is bounded by (16), (17) and our assumption that \((x_i)\) is dominated by \((x_{i,j})\). We also have that for any scalar \(\lambda\), \((T - \lambda I_{[(y_i)]-X})(y_k) = x_{n,k} - \lambda y_k\). But since \((y_k)\) is weakly null and \(x_{n,k} - \lambda y_k\) is not norm null, \(T - \lambda I_{[(y_i)]-X}\) is not compact. In other words \(T\) is not a compact perturbation of a scalar multiple of the inclusion. \(\square\)

Now we present the proof of Lemma 3.2. A less general version of this lemma can be found in [20, Lemma 2.4 (a) \(\Rightarrow\) (d)]. Schlumprecht assumes that the basic sequence \((z_i)\) is subsymmetric and satisfies Property P1 and we assume that the sequence \((x_i)\) has Property P2, which in view of Proposition 3.5 can be replaced by the assumption that \((z_i)\) has a spreading model which is unconditional and satisfies Property P1. Also Schlumprecht shows the result for some sequence \((\delta_n)\) while we show it for an arbitrary \((\delta_n)\). Additionally, we use different techniques than the ones used in [20]. Our arguments resemble the ones found in [2].

**Proof of Lemma 3.2.** Since \((x_n)\) has Property P2, for each \(\rho > 0\) we can define \(M = M(\rho)\) such that if \(\|\sum a_i x_i\| = 1\) then \(|\{i : |a_i| > \rho\}| \leq M\).

Let \((\varepsilon_j)_{j=1}^\infty \) be such that

\[
\sum_{j=2}^\infty \frac{\varepsilon_{j-1}}{\delta_j} \leq \frac{1}{2}
\]

Since \((z_n) \gg (\tilde{x}_n)\) by (3) we may choose a decreasing sequence \((\rho_j)_{j=1}^\infty \subseteq (0,1]\) such that \(\sum j \sqrt{\rho_j(j+1)} \leq 1/4\) and satisfying the following: for all \((a_i) \in c_00\) with \(|a_i| \in [0, \sqrt{\rho_j}]\) for each \(i\) and \(\|\sum a_i \tilde{x}_i\| = 1\) we have

\[
\|\sum a_i \tilde{x}_i\| \leq \varepsilon_j \|\sum a_i z_i\|
\]

Finally let \(M_j = M(\rho_j)\) as above.
By the definition of spreading models, by passing to a subsequence of \((x_i)\) and relabeling, we can assume that if \(j \leq F\) and \(|F| \leq M_j\) then for all \((a_i) \in c_0^0\),

\[
\frac{1}{2} \| \sum_{i \in F} a_i x_i \| \leq \| \sum_{i \in F} a_i \tilde{x}_i \| \leq 2 \| \sum_{i \in F} a_i x_i \|. \tag{19}
\]

Now fix \((a_i) \in c_0^0\) such that \(\| \sum a_i x_i \| = 1\). For \(j \in \mathbb{N}\) consider the vector \(\tilde{y} = \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} a_i \tilde{x}_i\). If \(\| \tilde{y} \| \geq \sqrt{\rho_j - 1}\) then

\[
\| \tilde{y} \| = \| \tilde{y} \| \frac{\tilde{y}}{\| \tilde{y} \|} \| \tilde{y} \| \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} \frac{a_i}{\| \tilde{y} \|} \| \tilde{x}_i \|
\leq \| \tilde{y} \| \varepsilon_{j-1} \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} \frac{a_i}{\| \tilde{y} \|} \| z_i \|
\leq \| \tilde{y} \| \varepsilon_{j-1} \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} a_i |z_i|.
\]

Thus in general, (without assuming that \(\| \tilde{y} \| \geq \sqrt{\rho_j - 1}\), we get

\[
\| \tilde{y} \| \leq \sqrt{\rho_j - 1} + \varepsilon_{j-1} \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} a_i |z_i|.
\tag{20}
\]

Let \(\rho_0\) be twice the basis constant of \((x_i)\) divided by the inf \(\| x_i \|\). Since \(\| \sum a_i x_i \| = 1\), we have that \(|a_i| \leq \rho_0\).

\[
1 = \| \sum a_i x_i \| \leq \sum_{j=1}^\infty \| \sum_{\rho_j < |a_i| \leq \rho_j - 1} a_i x_i \|
\leq \| \sum_{\rho_1 < |a_i| \leq \rho_0} a_i x_i \| + \sum_{j=2}^\infty \| \sum_{i \leq j : \rho_j < |a_i| \leq \rho_j - 1} a_i x_i \| + \sum_{j=2}^\infty \| \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} a_i x_i \|
\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \| \sum_{i \in F} a_i z_i \| + \sum_{j=2}^\infty j \rho_j - 1 + 2 \sum_{j=2}^\infty \| \sum_{i > j : \rho_j < |a_i| \leq \rho_j - 1} a_i \tilde{x}_i \|
\]

where

\[
\delta_1 = \sup \left\{ \frac{\| \sum_{i \in F} a_i x_i \|}{\| \sum_{i \in F} a_i z_i \|} : (a_i)_{i \in F} \subseteq \mathbb{C} \text{ with } |F| \leq M_1 \text{ and } (a_i)_{i \in F} \neq 0^F \right\}
\]

which is clearly finite by using an \(\ell_1\) estimate for the numerator and an \(\ell_\infty\) estimate for the denominator. Note the third piece of the last inequality is true by (19) since the cardinality
of \( \{i > j : \rho_j < |a_i| \leq \rho_{j-1}\} \) is at most \( M_j \). Continuing the calculations from above, we get

\[
1 \leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \| \sum_{i \in F} a_i z_i \| + \frac{1}{4} + 2 \sum_{j=2}^{\infty} \sqrt{\rho_{j-1}} + 2 \sum_{j=2}^{\infty} \varepsilon_{j-1} \| \sum_{i > j : \rho_j < |a_i| \leq \rho_{j-1}} a_i z_i \| \quad \text{by (20)}
\]

\[
\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \| \sum_{i \in F} a_i z_i \| + \frac{1}{2} + 2 \sum_{j=2}^{\infty} \varepsilon_{j-1} \| \sum_{i > j : \rho_j < |a_i| \leq \rho_{j-1}} a_i z_i \| + \sup_{n \geq 2} \frac{\delta_n}{|F| \leq M_n} \| \sum_{i \in F} a_i z_i \|.
\]

Thus

\[
1 \leq 2 \sup_{n \in \mathbb{N}} \sup_{n \leq F \subseteq \mathbb{N}, |F| \leq M_n} \delta_n \| \sum_{i \in F} a_i z_i \|
\]

proving the lemma. \( \square \)

Now we present the proof of Lemma 3.3.

**Proof of Lemma 3.3.** Assume that for each \( n \in \mathbb{N} \) \((x_{j}^{M_n})_j\) has spreading model \((\tilde{x}_{j}^{M_n})_j\), \((\tilde{x}_{j}^{M_n})_j\) \(\varepsilon\)-dominates \((z_i)_i=1\) and moreover if \(|F| \leq M_n\) and \((a_j)_{j \in F}\) are scalars then

\[
(21) \quad \frac{1}{2} \| \sum_{j \in F} a_j \tilde{x}_{j}^{M_n} \| \leq \| \sum_{j \in F} a_j x_{j}^{M_n} \| \leq 2 \| \sum_{j \in F} a_j \tilde{x}_{j}^{M_n} \|.
\]

By Remark 2.3 by passing to subsequences and relabeling, assume that \((x_{j}^{M_n})_{(n,j) \in I}\) forms a regular array with basis constant at most equal to 2. Apply Theorem 2.9 to \((x_{j}^{M_n})_{(n,j) \in I}\) to get a subarray which satisfies the conclusion of Theorem 2.9. By relabeling call \((x_{j}^{M_n})_{(n,j) \in I}\) the resulting subarray. Define

\[
y_j = \sum_{n=1}^{j} \delta_n x_{\ell_j}^{M_n}
\]

where \((\ell_j)\) is an increasing sequence of positive integers which guarantees that \(|x_i^* y_j| < \varepsilon\).

Note that \((y_j)\) is weakly null since \((x_{j}^{M_n})_j\) is weakly null for all \( n \) and \((\delta_n)\) is summable. Since \((x_{j}^{M_n})_{(n,j) \in I}\) is regular, \((y_j)\) is a basic sequence with

\[
(22) \quad \| y_j \| \geq \frac{\delta_1}{2} \inf_{n,m} \| x_m^n \|
\]

(since the basis constant of \((x_{j}^{M_n})_{(n,j) \in I}\) is at most equal to 2 by Remark 2.3). Since \((\delta_n)\) is summable, \((y_j)\) is also bounded. Fix \( n \in \mathbb{N} \) and let \( n \leq F \subseteq \mathbb{N} \), with \(|F| \leq M_n\). Then
\[ \| \sum a_j y_j \| \geq \frac{1}{3} \| \sum_j \delta_n a_j x_j^{M_n} \| \text{ (by Theorem 2.9)} \]
\[ \geq \frac{1}{6} \| \sum_{j \in F, j \geq n} \delta_n a_j \tilde{x}_j^{M_n} \| \text{ (by (21))} \]
\[ = \frac{1}{6} \| \sum_{j \in F, j \geq n} \delta_n a_j \tilde{x}_j^{M_n} \| \]
where the map \( F \ni j \mapsto k_j \in \{1, 2, \ldots, |F|\} \) is a 1-1 increasing function.

\[ \geq \frac{1}{6} C \delta_n \| \sum_{j \in F, j \geq n} a_j \tilde{z}_j \| \text{ (by (6))} \]
Thus
\[ \| \sum a_j y_j \| \geq \sup_n \frac{1}{6C} \sup_{n \leq |F| \leq |F| \leq M_n} \delta_n \| \sum_{i \in F} a_i z_i \|. \]

Theorem 3.1, just as Theorem 0.7, gives sufficient conditions on a Banach space \( X \), in order that the “multiple of the inclusion plus compact” problem has an affirmative solution on \( X \). If in Theorem 3.1 one considers the special case where \( (x_i)_i = (x_i)_i \), then the assumptions of Theorem 3.1 are similar to the assumptions of Theorem 0.7. The difference in that case is that in Theorem 3.1 (but not in Theorem 0.7), we assume that the sequence \( (x_i)_i \) satisfies Property P2. Instead, in Theorem 0.7 we assume that the sequence \( (z_i)_i \) has a spreading model which has Property P2. In Proposition 3.4 we show that if the basic sequence \( (z_i)_i \) has a spreading model which has Property P2 then by replacing \( (z_i)_i \) by a new sequence and relabeling, we can assume that \( (z_i)_i \) has a spreading model which is unconditional and has Property P1. Then, in Proposition 3.5 we show that if the basic sequence \( (z_i)_i \) has a spreading model which is unconditional and has Property P1 then the sequence \( (x_i)_i \) can be “replaced” by a sequence \( (x_i^+) \) which (may not be contained in the Banach space \( X \) and) satisfies Property P2. Thus the proof of Theorem 0.7 will follow from Theorem 3.1 and Propositions 3.4 and 3.5.

**Proposition 3.4.** Let \( (z_i)_i \) be a seminormalized basic sequence which has a spreading model which has Property P2. Then there exists a subsequence \( (z_{k_n})_i \) of \( (z_i)_i \) such that \( (Z_i)_i \) has a spreading model which is unconditional and has Property P1, where either \( (Z_i)_i := (z_{k_n})_i \) or \( (Z_i)_i := (z_{k_n} - z_{k_{n-1}})_i \).

**Proof.** By Rosenthal’s \( \ell_1 \) Theorem [17] there exists a subsequence \( (z_{k_n})_i \) of \( (z_i)_i \) such that either \( (z_{k_n})_i \) is equivalent to the standard basis of \( \ell_1 \), or \( (z_{k_n})_i \) is weak Cauchy. In the first case by passing to a further subsequence and relabeling assume that \( (z_{k_n})_i \) has a spreading model and set \( (Z_i)_i := (z_{k_n})_i \). Then obviously \( (Z_i)_i \) has a spreading model which is unconditional and has Property P1. If \( (z_{k_n})_i \) is weak Cauchy, set \( (Z_i)_i := (z_{k_n} - z_{k_{n-1}})_i \). Then \( (Z_i)_i \) is weakly null, hence by [5], [6] we can pass to a subsequence of \( (Z_i)_i \) and relabel in order to assume that \( (Z_i)_i \) has a suppression 1-unconditional spreading model. It is obvious to see that the Property P2 passes from the spreading model of \( (z_i)_i \) to the spreading model of \( (Z_i)_i \). Thus \( (Z_i)_i \) has a spreading model which is unconditional and has Property P1 (see Proposition 3.6(b)). \( \square \)
Proposition 3.5. Let \((x_i)\) and \((z_i)\) be two seminormalized basic sequences (not necessarily in the same Banach space) such that \((x_i)\) satisfies condition (5) of Theorem 0.7 and \((z_i)\) has a spreading model which is unconditional and has Property P1. Then there exists a seminormalized basic sequence \((x_i^+)\) which has Property P2, dominates \((x_i)\) and has spreading model \((\tilde{x}_i^+)\) which is s.c. dominated by \((z_i)\).

Proof of Proposition 3.5. Before defining \((x_i^+)\), define an auxiliary basic sequence \((z_i')\) as follows. Define the norm on the span of \((z_i')\) as the completion of the following: for \((a_i) \in c_{00}\) let

\[
\| \sum a_i z_i' \| := \sup_{n \in \mathbb{N}} \left\{ \frac{1}{\sqrt{L_n}} \| \sum_{i \in A} a_i \tilde{z}_i \| : A \subseteq \mathbb{N} \text{ with } |A| \leq n \right\}
\]

where

\[
L_n = \sup_{1 \leq k \leq n} \| \sum_{i=1}^{k} \tilde{z}_i \|.
\]

Claim 1: The sequence \((z_i')\) is seminormalized, 1-spreading, unconditional (thus 1-subsymmetric) and has Property P1.

Since \((\tilde{z}_i)\) is 1-spreading, the Property P1 of \((\tilde{z}_i)\) is equivalent to the fact that \(L_n \to \infty\). Also it is easy to verify that the 1-spreading and unconditionality properties pass from \((\tilde{z}_i)\) to \((z_i')\). Since \((L_n)\) is increasing we have that \(\| z_i' \| = \| \tilde{z}_i \| = \sqrt{L_1} \| \tilde{z}_i \|\) therefore \((z_i')\) is seminormalized. Notice that

\[
L_n = \sup_{1 \leq k \leq n} \| \sum_{i=1}^{k} \tilde{z}_i \| \leq C_1 \| \sum_{i=1}^{n} \tilde{z}_i \|
\]

where \(C_1\) is the basis constant of \((\tilde{z}_i)\). Thus

\[
\| \sum_{i=1}^{n} z_i' \| \geq \frac{1}{\sqrt{L_n}} \| \sum_{i=1}^{n} \tilde{z}_i \| \geq \frac{1}{C_1} \sqrt{L_n} \to \infty.
\]

Hence \((z_i')\) has Property P1 (since \((z_i')\) is 1-spreading). This finishes the proof of Claim 1.

Claim 2: \((z_i) \gg (z_i')\).

Let \(\varepsilon > 0\) be given. We will choose \(\rho > 0\) so that \(\Delta_{(z_i), (z_i')}(\rho) < \varepsilon\). Since \(L_n \to \infty\) we can find an \(N \in \mathbb{N}\) such that \(C_2/\sqrt{L_N} < \varepsilon\) where \(C_2\) is the suppression unconditionality constant of the sequence \((\tilde{z}_i)\) (by [5, 6] we have \(C_2 = 1\) if \((z_i)\) is weakly null). Let \(\rho = \min_{n \leq N} \frac{\varepsilon \sqrt{L_n}}{\| \tilde{z}_i \| n}\). Let \((a_i) \in c_{00}\) be such that \(\| \sum a_i z_i \| = 1\) and \(|a_i| \leq \rho\). Also let \(n_0 \in \mathbb{N}\) and \(A\) be a subset of \(\mathbb{N}\) with \(|A| \leq n_0\) and \(\| \sum a_i z_i' \| = \frac{1}{\sqrt{L_{n_0}}} \| \sum_{i \in A} a_i \tilde{z}_i \|\). If \(n_0 \leq N\) then

\[
\frac{1}{\sqrt{L_{n_0}}} \| \sum_{i \in A} a_i \tilde{z}_i \| \leq \frac{\| \tilde{z}_i \|}{\sqrt{L_{n_0}}} \rho n_0 \leq \varepsilon
\]

by the choice of \(\rho\) (notice that \(\| \tilde{z}_i \| = \| \tilde{z}_1 \|\) for all \(i\)). If \(n_0 > N\) then

\[
\frac{1}{\sqrt{L_{n_0}}} \| \sum_{i \in A} a_i \tilde{z}_i \| \leq \frac{1}{\sqrt{L_N}} C_2 \| \sum a_i \tilde{z}_i \| = \frac{C_2}{\sqrt{L_N}} < \varepsilon
\]

where the first inequality follows by the unconditionality of \((\tilde{z}_i)\). Thus \(\Delta_{(z_i), (z_i')}(\rho) < \varepsilon\). This finishes the proof of Claim 2.
Now we are ready to define the basic sequence \((x_i^+)\). Define the norm on the span of the basic sequence \((x_i^+)\) as the completion of the following: for \((a_i) \in c_{00}\) let
\[
\| \sum a_i x_i^+ \| = \max \{ \| \sum a_i x_i \|, \| \sum a_i z_i' \| \}
\]
where \((z_i')\) is defined by (23).

**Claim 3:** The sequence \((x_i^+)\) is seminormalized, dominates \((x_i)\), has Property P2 and has spreading model \((\bar{x}_i^+)\) which satisfies \((z_i) >> (\bar{x}_i^+)\).

Indeed by Claim 1 we have that \((z_i')\) is seminormalized thus \((x_i^+)\) is seminormalized. Obviously, by (25), we have that \((x_i^+)\) dominates \((x_i)\). By Claim 1 we have that \((z_i')\) is unconditional and has Property P1 thus it is easy to see that \((z_i')\) has Property P2, (see Proposition 3.6 (c)). Since \((x_i^+)\) dominates \((z_i')\), we obtain that \((x_i^+)\) has Property P2. Since \((x_i)\) has spreading model \((\bar{x}_i)\) and \((z_i')\) is 1-spreading, (25) implies that \((x_i^+)\) has spreading model \((\bar{x}_i^+)\) which satisfies
\[
\| \sum a_i \bar{x}_i^+ \| = \max \{ \| \sum a_i \bar{x}_i \|, \| \sum a_i z_i' \| \}.
\]

Finally, Claim 2, the fact that \((z_i) >> (\bar{x}_i)\) and (26) imply that \((z_i) >> (\bar{x}_i^+)\). This finishes the proof of Claim 3 and of Proposition 3.5.

**Proof of Theorem 0.7.** We know that \((z_i)\) has a spreading model which has Property P2. Then by Proposition 3.4, there exists a subsequence \((z_{k_i})\) of \((z_i)\) such that \((Z_{i})\) has a spreading model which is unconditional and has Property P1, where either \((Z_{i})_i := (z_{k_i})_i\) or \((Z_{i})_i := (z_{k_{2i}} - z_{k_{2i-1}})_i\).

First assume that \((Z_{i})_i = (z_{k_i})_i\). Then set \((X_i)_i := (x_i)_i\) and \((X_{n})_i := (x_{k_i})_i\). We claim that (5) and (6) are satisfied with “\((x_i)\)”,” \((x_i^+)\)” and “\((z_i)\)” being replaced by \((X_i)\), \((X_{n})_i\) and \((Z_{i})_i\) respectively. Indeed, since \((\bar{x}_i) << (z_i)\), we have that \((\bar{x}_i) << (Z_{i})_i\). Since \((\bar{x}_i)\) is isometrically equivalent to \((x_i)\), we obtain that \((\bar{x}_i) << (Z_{i})_i\). Thus (5) is satisfied for \((X_i)_i\) and \((Z_{i})_i\). Also notice that (6) is satisfied with “\((x_{n})\)” and “\((z_i)\)” being replaced by \((X_{n})_i\) and \((Z_{i})_i\) respectively. Indeed, since \((z_{i})_{n-1} = C\)-dominated by \((\bar{x}_{n})_i\), we have that \((Z_{i})_{n-1} = C\)-dominated by \((\bar{x}_{k_{n}})_i\) which is isometrically equivalent to \((\bar{x}_{k_{n}})_i\).

Second assume that \((Z_{i})_i := (z_{k_{2i}} - z_{k_{2i-1}})_i\). Then set \((X_i)_i := (x_{2i} - x_{2i-1})_i\) and \((X_{n})_i := (x_{k_{2i}} - x_{k_{2i-1}})_i\). We claim that (5) and (6) are satisfied for “\((x_i)\)”,” \((x_{n})\)” and “\((z_i)\)” being replaced by \((X_i)\), \((X_{n})_i\) and \((Z_{i})_i\) respectively. Indeed notice that since \((\bar{x}_i) << (z_i)\), we have that \((\bar{x}_{k_{2i}} - \bar{x}_{k_{2i-1}}) << (Z_{i})_i\). Since \((\bar{x}_{k_{2i}} - \bar{x}_{k_{2i-1}})\) is isometrically equivalent to \((\bar{x}_{2i} - \bar{x}_{2i-1})\) and \((X_i)\) has spreading model isometrically equivalent to \((\bar{x}_{2i} - \bar{x}_{2i-1})\), we have that (5) is satisfied for \((X_i)\) and \((Z_{i})_i\). Also, since \((z_{i})_{n-1} = C\)-dominated by \((\bar{x}_{n})_i\), we have that \((Z_{i})_{n-1} = C\)-dominated by \((\bar{x}_{k_{2n}} - \bar{x}_{k_{2n-1}})_i\), which is isometrically equivalent to \((\bar{x}_{2i} - \bar{x}_{2i-1})\). Finally notice that the spreading model of \((X_{n})_i\), is isometrically equivalent to \((\bar{x}_{k_{2n}} - \bar{x}_{k_{2n-1}})_i\). Therefore condition (6) is satisfied for \((X_{n})_i\) and \((Z_{i})_i\).

In either of the above two cases (i.e. either \((Z_{i})_i := (z_{k_i})_i\) or \((Z_{i})_i := (z_{k_{2i}} - z_{k_{2i-1}})_i\)), since \((\bar{x}_i) << (Z_{i})_i\) and \((Z_{i})_i\) has a spreading model which is unconditional and has Property P1, by Proposition 3.5, there exists a seminormalized basic sequence \((x_i^+)\) which has Property P2, dominates \((X_i)\) and has spreading model \((\bar{x}_i^+)\) which is s.c. dominated by \((Z_{i})_i\). Apply
Theorem 3.1 for “$(\bar{x}_i)$,” being equal to $(X_i)$, “$(x^n_i)$” being equal to $(X^n_i)$, “$(z_i)$” being equal to “$(Z_i)$” and “$(x_i)_i$” being equal to $(x^+_i)$ to finish the proof. \□

Proof of Theorem 0.8. If $p$ belongs to the Krivine set of $(\bar{x}_i)$ then for all $n \in \mathbb{N}$ there exists $(x^n_i)_{i \in \mathbb{N}}$ a block sequence of $(x_i)$ of identically distributed blocks such that any $n$ terms of $(x^n_i)_{i \in \mathbb{N}}$ are 2-equivalent to the unit vector basis of $\ell^n_p$. Then apply Theorem 0.7 for $(z_i)$ being the unit vector basis of $\ell_p$. \□

Next we examine the relation between Properties P1 and P2 and how these properties pass to spreading models.

Proposition 3.6. (a) Assume that a seminormalized basic sequence $(z_i)$ has a spreading model $(\bar{z}_i)$. If $(z_i)$ has Property P1 then $(\bar{z}_i)$ has Property P1. If $(z_i)$ has Property P2 then $(\bar{z}_i)$ has Property P2.

(b) If a seminormalized basic sequence has Property P2 then it has Property P1.

(c) If an unconditional seminormalized basic sequence has Property P1 then it has Property P2.

(d) Let $(z_i)$ be a Schreier unconditional seminormalized basic sequence in a Banach space. If $(z_i)$ has Property P1 then there exists some subsequence $((z_{m_i}))$ of $(z_i)$ which has Property P2.

Proof. Parts (a), (b) and (c) are obvious. For part (d) assume (towards contradiction) that no subsequence of $(z_i)$ has Property P2. So for every subsequence $(m_i)$ of $\mathbb{N}$ there exists $\rho > 0$ such that for all $M \in \mathbb{N}$ there exists $(a_i) \in c_{00}$ with $\|\sum a_i z_{m_i}\| = 1$ and $|\{i : |a_i| > \rho\}| > M$. Thus $\rho$ depends on the sequence $(m_i)$. Let $\rho((m_i))$ be the supremum of such $\rho$’s. Notice that if $(m_i), (n_i)$ are subsequences of $\mathbb{N}$ with $(n_i) \subseteq (m_i)$ then $\rho((n_i)) \leq \rho((m_i))$. We claim there is a subsequence $(m_i)$ of $\mathbb{N}$ such that $\inf\{\rho((m_{n_i})) : (n_i) \text{ subsequence of } \mathbb{N}\} > 0$. To show this claim we assume again by contradiction that for all subsequences $(m_i)$ of $\mathbb{N}$ we have $\inf\{\rho((m_{n_i})) : (n_i) \text{ subsequence of } \mathbb{N}\} = 0$.

Let $(n^1_i)_{i=1}^\infty \subseteq \mathbb{N}$ be such that $\rho((n^1_i)) < 1$. Let $(n^2_i)_{i=1}^\infty \subseteq (n^1_i)_{i=1}^\infty$ be such that $\rho((n^2_i)) < \frac{\rho((n^1_i))}{2C_s}$, where $C_s$ is the constant of Schreier unconditionality of $(z_i)$. Given $(n^k_i)_{i=1}^\infty$ define $(n^{k+1}_i)_{i=1}^\infty \subseteq (n^k_i)_{i=1}^\infty$ to be a subsequence such that $\rho((n^{k+1}_i)) < \frac{\rho((n^k_i))}{2C_s}$. Define $n_i := n^k_i$. By our assumption $\rho((n_i)) > 0$. Let $k \in \mathbb{N}$ be such that $\rho((n_i)) > \rho((n^k_i)) > 0$. Let $N \in \mathbb{N}$ be arbitrary such that $N > k$. By the definition of $\rho((n_i))$ there exist a sequence of scalars $(a_{n_i})$ such that $\|\sum a_{n_i} z_{n_i}\| = 1$ and $|\{n_i : |a_{n_i}| > \rho((n_i))\}| > 2(N + 1)$. Define $A := \{n_i : |a_{n_i}| > \frac{\rho((n_i))}{2}\}$ and $B = \{\ell_{m+1}, \ell_{m+2}, \ldots, \ell_{|A|}\}$. Notice that $B$ is a Schreier subset of the sequence $(n_i^{k+1})$ and $|B| \geq N$. Thus we can project to the set $B$:

$$\|\sum_{n_i \in B} a_{n_i} z_{n_i}\| \leq C_s \|\sum a_{n_i} z_{n_i}\| = C_s.$$

Thus $\|\sum_{n_i \in B} a_{n_i} z_{n_i}\| \leq 1$. Note that $B \subseteq \{n_i : |a_{n_i}| > \frac{\rho((n_i))}{2C_s}\}$. Since $B \subset (n_i^{k+1})$ we have that $\rho((n_i^{k+1})) \geq \frac{\rho((n_i))}{2C_s}$. But $\rho((n_i)) > \rho((n^k_i)) > 2C_s \rho((n_i^{k+1}))$, a contradiction. Thus there exists some subsequence $(n_i)$ where $\inf\{\rho((m_{n_i})) : (m_i) \text{ subsequence of } \mathbb{N}\} > 0$. 23
For ease of notation call \((z_i)\) the subsequence \((z_{n_i})\). Let \(\rho > 0\) such that

for all sequences \((s_i)\) of positive integers and \(m \in \mathbb{N}\) there exists

\[
(a_i) \in c_{00} \text{ with } \| \sum a_{s_i} z_{s_i} \| = 1 \text{ and } |\{ i : |a_{s_i}| > \rho \}| > m.
\]

Fix \(m \in \mathbb{N}\). Let

\[
\sigma_m = \{(\ell_i)_{i=1}^m \subseteq \mathbb{N} : \text{ there exists } (a_i) \in c_{00} \text{ such that } \| \sum a_i z_i \| = 1 \text{ and } (\ell_i)_{i=1}^m \subseteq \{ i : |a_i| > \rho \}\}.
\]

By Ramsey’s theory there exists some \((m_i)_{i=1}^\infty \subseteq \mathbb{N}\) such that either \([(m_i)_{i=1}^\infty]^{m} \subseteq \sigma_m\) or \([(m_i)_{i=1}^\infty]\) \(\cap \sigma_m = \emptyset\). But the second case is not possible by (27).

Thus for each \(m \in \mathbb{N}\) there exists some subsequence \((n_i^m)\) of \(\mathbb{N}\) such that for each \(F \subseteq (n_i^m)\) where \(|F| = m\), there exists some \((a_i) \in c_{00}\) such that \(\| \sum a_i z_i \| = 1\) and \(F \subseteq \{ i : |a_i| > \rho \}\).

We can choose each of these subsequences such that \((n_i^1) \supseteq (n_i^2) \supseteq (n_i^3) \supseteq \cdots\) and then define \(n_i = n_i^i\). Thus for all \(A \subseteq \{m, m+1, \ldots\}\) such that \(|A| = m\) (i.e. \(A\) is Schreier) there exists some \((a^m_i) \in c_{00}\) such that \(\| \sum a^m_i z_i \| = 1\) and \((n_i)_{i \in A} \subseteq \{ i : |a^m_i| > \rho \}\). So we define for each \(m \in \mathbb{N}\) such \((a^m_i)\) where \(\| \sum a_i^m z_i \| = 1\) and \((n_i)_{i=m}^{2m-1} \subseteq \{ i : |a^m_i| > \rho \}\). Notice each \(a_i^m \in A := \{ z \in \mathbb{C} : \rho < |z| < \frac{2C}{\inf \| z_i \|}\}\) where \(C\) is the basis constant of \((z_i)\). It is an elementary exercise to see the following.

For all \(M \in \mathbb{N}\) and \(\varepsilon > 0\) there exists \(k \in \mathbb{N}\) such that \((a_i)_{i=1}^k \subseteq A\)

\[
(28)
\]

then there exists some subset \(A' \subseteq \{1, 2, \ldots, k\}\) such that

\[|A'| \geq M\]

and for all \(i, j \in A'\) we have \(|a_i - a_j| < \varepsilon\).

Since \((z_i)\) has Property P1,

\[
\lim \inf_{n \to \infty} \inf_{A \subseteq \mathbb{N} : |A| = n} \| \sum_{i \in A} z_i \| = \infty.
\]

So there exists \(N \in \mathbb{N}\) such that for all sets \(A \subseteq \mathbb{N}\) with \(|A| = N\) we have

\[
(29)
\]

\[
\| \sum_{i \in A} z_i \| > \frac{C_s}{\rho}.
\]

By (28) for \(\varepsilon = \frac{C_s}{2N}\) and \(M = N\) to get \(k \in \mathbb{N}\). Then note that \((a_i^k)_{i=k}^{2k-1} \subseteq A\) has \(k\) many terms (some terms may be equal). Therefore there is an \(N\) element set \(A' \subseteq \{k, k+1, \ldots, 2k - 1\}\) such that for all \(i, j \in A'\) we have \(|a_i^k - a_j^k| < \varepsilon\). Notice that \(A'\) is a Schreier set. Thus by Schreier unconditionality we have

\[
C_s = \| \sum a_i^k z_i \| \geq \| \sum a_i^k z_i \| \geq \| \sum a_i^k z_i \| - \| \sum (a_i^k - a_j^k) z_i \|
\]

\[
\geq \| \sum a_i^k z_i \| - N \max |a_i^k - a_j^k| \geq \| \sum a_i^k z_i \| - \frac{C_s}{2}.
\]

Therefore \(\| \sum_{i \in A'} z_i \| \leq \frac{C_s}{2a_i^k} \leq \frac{C_s}{2\rho}\) which is a contradiction to (29). \(\square\)
Note that the summing basis has Property P1 but no subsequence of it has Property P2. Thus the assumption that \((z_i)\) is Schreier unconditional in Proposition 3.6(d) is needed.

Notice that if \((x_i), (x_i^n)\) and \((z_i)\) are seminormalized basic sequences satisfying conditions (5) and (6) of Theorem 0.7, and \((z_k_i)\) is any subsequence of \((z_i)\) then \((x_i), (x_i^{k_n})\) and \((z_k_i)\) also satisfy conditions (5) and (6). Indeed, the sequence \((\tilde{x}_i)\) is isometrically equivalent to the sequence \((\tilde{x}_k)\) which is s.c. dominated by \((z_k_i)\). Also \((z_k_i)\) is \(C\)-dominated by \((\tilde{x}_k^n)_{n=1}^{\infty}\) which is isometrically equivalent to \((\tilde{x}_i^n)_{n=1}^{\infty}\). This observation, Theorem 0.7 and Proposition 3.6(a) and (c), immediately give the following result.

**Corollary 3.7.** Let \(X\) be a Banach space containing seminormalized basic sequences \((x_i)\) and \((x_i^n)\) for all \(n \in \mathbb{N}\), such that \(0 < \inf_{n,i} \|x_i^n\| \leq \sup_{n,i} \|x_i^n\| < \infty\). Let \((z_i)\) be a basic sequence not necessarily in \(X\). Assume that \((x_i)\) satisfies condition (5) and for all \(n \in \mathbb{N}\) the sequence \((x_i^n)\) satisfies condition (6) of Theorem 0.7. Assume that the sequence \((z_i)\) satisfies at least one of the following three conditions:

- a subsequence of \((z_i)\) has a spreading model which has Property P2, or
- a subsequence of \((z_i)\) has Property P1 and has a spreading model which is unconditional, or
- a subsequence of \((z_i)\) has a spreading model which is unconditional and has Property P1, or
- a subsequence of \((z_i)\) has Property P2.

Then there exists a subspace \(Y\) of \(X\) which has a basis and an operator \(T \in \mathcal{L}(Y, X)\) which is not a compact perturbation of a multiple of the inclusion map.

4. An Application of Theorem 0.7

Next we give an application of Theorem 0.8. As mentioned before the problem of finding a subspace \(Y\) of a Banach space \(X\) and an operator \(T \in \mathcal{L}(Y, X)\) which is not a compact perturbation of the inclusion operator is non trivial when \(X\) is saturated with HI Banach spaces. The HI space to which Theorem 0.8 will be applied was constructed by N. Dew [7], and here will be denoted by \(D\). The construction of the space \(D\) is based on the 2-convexification of the Schlumprecht space \(S\) [19] in a similar manner that the space of T.W. Gowers and B. Maurey [11] is based on \(S\). We recall the necessary definitions.

Let \(X\) be a Banach space with a basis \((e_i)\). For any interval \(E \in \mathbb{N}\) and a vector \(x = \sum x_je_j \in X\) define \(Ex = \sum_{j \in E} x_je_j \in X\). There is a unique norm \(\|\cdot\|_S\) on \(c_00\) which satisfies:

\[
\|x\|_S = \sup \left\{ \frac{1}{f(\ell)} \sum_{i=1}^\ell \|E_ix\|_S : E_1 < \cdots < E_\ell \right\} \vee \|x\|_\ell_\infty
\]

where \(f(\ell) = \log_2(\ell + 1)\). The completion of \(c_00\) under this norm is the Banach space \(S\). Let \(S_2\) be its 2-convexification. Recall if \(X\) is a Banach space having an unconditional basis \((e_i)\), then we can define the 2-convexification \(X_2\) of \(X\) by the norm

\[
\|\sum a_ne_n\|_{X_2} := \left(\sum a_n^2 e_n \|X\|_{X_2}\right)^{1/2}
\]

where \((\sqrt{e_n})\) denotes the basis of \(X_2\), (we will talk more later about the “square root” map from \(X\) to \(X_2\)). We will show that the spreading model of the unit vector basis of \(D\) is the unit vector basis of \(S_2\). Before we do this we must see the definition of \(D\).
In order to define the Banach space $D$, a lacunary set $J \subseteq \mathbb{N}$ is used which has the property that if $n, m \in J$ and $n < m$ then $8n^4 \leq \log \log m$, and $f(\min J) \geq 45^4$. Write $J$ in increasing order as $\{j_1, j_2, \ldots\}$. Now let $K \subseteq J$ be the set $\{j_1, j_3, j_5, \cdots\}$ and $L \subseteq J$ be the set $\{j_2, j_4, j_6, \ldots\}$. Let $Q$ be the set of scalar sequences with finite support and rational coordinates whose absolute value is at most one. Let $\sigma$ be an injective function from $Q$ to $L$ such that if $z_1, \ldots, z_i$ is such a sequence, then $(1/400)f(\sigma(z_1, \ldots, z_i)^{1/40})^{\frac{1}{3}} \geq \#\text{supp}(\sum_{j=1}^i z_j)$. Then, recursively, we define a set of functionals of the unit ball of the dual space $D^*$ as follows: Let

$$D_0^* = \{\lambda e_n^* : n \in \mathbb{N}, |\lambda| \leq 1\}.$$  

Assume that $D_k^*$ has been defined. Define the norm $\| \cdot \|_k$ on $c_{00}$ by

$$\|x\|_k = \sup\{|x^*(x)| : x^* \in D_k^*\}$$

and let $\| \cdot \|_k^*$ denote its dual norm. Then $D_{k+1}^*$ is the set of all functionals of the form $F(z)^*$ where $F \subseteq \mathbb{N}$ is an interval and $z^*$ has one of the following three forms:

$$z^* = \sum_{i=1}^\ell \alpha_i z_i^*$$

where $\sum_{i=1}^\ell |\alpha_i| \leq 1$ and $z_i^* \in D_k^*$ for $i = 1, \ldots, \ell$.

$$z^* = \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^\ell \alpha_i z_i^*$$

where $\sum_{i=1}^\ell \alpha_i^2 \leq 1$, $z_i^* \in D_k^*$ for $i = 1, \ldots, \ell$, and $z_1^* < \cdots < z_\ell^*$.

$$z^* = \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^\ell z_i^* \text{ where } z_i^* = \frac{1}{\sqrt{f(m_i)}} \sum_{j=1}^{m_i} \alpha_{i,j} \frac{Ez_{i,j}^*}{\|Ez_{i,j}^*\|_k}$$

for certain $(\alpha_{i,j})$ where $\sum_{i,j} \alpha_{i,j}^2 \leq 1$ (these values are explicitly chosen in [7], but the exact values are not needed for our purposes) where $z_{i,j}^* \in D_k^*$ for $1 \leq i \leq \ell$ and $1 \leq j \leq m_i$, $z_{1,1}^* < \cdots < z_{1,m_1}^* < z_{2,1}^* < \cdots < z_{\ell,m_\ell}^*$, $m_1 = j_2$, $\beta z_i^*$ has rational coordinates for some $\beta > 0$ (whose exact value is not needed for our purpose), $m_{i+1} = \sigma(\beta z_1^*, \ldots, \beta z_i^*)$, for $i = 1, \ldots, \ell-1$ and $E$ is an interval.

Finally, the norm of $D$ is defined by

$$\|x\|_D = \sup\{z^*(x) : z^* \in \bigcup_{k=0}^\infty D_k^*\}.$$  

**Proposition 4.1.** The spreading model of the unit vector basis of $D$ is the unit vector basis of $S_2$.

**Proof.** From the definitions of the two spaces it is easy to see that for $(a_i) \in c_{00}$, $\| \sum a_i e_i \|_{S_2} \leq \| \sum a_i e_i \|_D$ (($e_n$) will denote the bases of both spaces $S_2$ and $D$ but there will be no confusion about which space we consider at each moment). Thus to show Proposition 4.1 we need only show for any given $\epsilon > 0$, and finitely many scalars $(a_i)_{i=1}^N$, there exists $n_0$ such that for any $n_1, n_2, \ldots, n_N \in \mathbb{N}$ with $n_0 < n_1 < n_2 < \cdots < n_N$, we have

$$\|x\|_D \leq \|y\|_{S_2} + \epsilon.$$
where \( x = \sum_{i=1}^{N} a_i e_{n_i} \in D \) and \( y = \sum_{i=1}^{N} a_i e_i \in S_2 \). This will follow immediately once we show by induction on \( n \) that for any \( n \in \mathbb{N} \), \( \varepsilon > 0 \) and scalars \((a_i)_{i=1}^{N}\) we have

\[
\|x\|_n \leq \|y\|_{S_2} + \varepsilon
\]

(34)

where \( \| \cdot \|_n \) is defined in (30). For \( “n = 0” \) we have \( \|x\|_0 = \max_{1 \leq i \leq N} |a_i| \leq \|y\|_{S_2} \). Now for the inductive step assume that (34) is valid for \( n \). Let \( \varepsilon > 0 \) and scalars \((a_i)_{i=1}^{N}\).

First note that by the induction hypothesis there exists \( n_1^0 \in \mathbb{N} \) such that for all \( n_1^0 < n_1 < n_2 < \cdots < n_N \) we have

\[
\|x\|_n \leq \|y\|_{S_2} + \varepsilon.
\]

(35)

Secondly, by the inductive hypothesis there exists \( n_2^0 \in \mathbb{N} \) such that for all \( 1 \leq i_0 \leq j_0 \leq N \) and \( n_2^0 < n_1 < n_2 < \cdots < n_N \) we have

\[
\| \sum_{i=1}^{j_0} a_i e_{n_i} \|_n \leq \| \sum_{i=i_0}^{j_0} a_i e_i \|_{S_2} + \frac{\varepsilon}{\sqrt{N}}.
\]

(36)

And finally there exists \( j_0 \in L \) such that \( \frac{1}{\sqrt{f(j_0)}} N \|y\|_{S_2} \leq \varepsilon \). Let \( G = \sigma^{-1}(\{1, 2, \ldots, j_0 - 1\}) \), So \( G \) is a finite subset of finite sequences of vectors with rational coefficients. Let \( n_0^3 \) be the maximum of the support of any vector in any sequence in \( G \). And then set \( n_0 = \max\{n_0^1, n_0^2, n_0^3\} \). Recall the norming vectors \( z^* \in D_{n+1}^* \) can be one of three different types. Each type of functional will present us with a different case.

**CASE 1:** Let \( z^* \) be given by (31). Then

\[
|z^*(x)| \leq \sum_{i=1}^{\ell} |\alpha_i| |z_i^*(x)|
\]

\[
\leq \sum_{i=1}^{\ell} |\alpha_i| (\|x\|_{S_2} + \varepsilon) \text{ (by the (35))}
\]

\[
= \|x\|_{S_2} + \varepsilon.
\]

**CASE 2:** Let \( z^* \) be given by (32). Thus for \( n_0 < n_1 < n_2 < \cdots < n_N \) we have

\[
|z^*(x)| \leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| |z_j^*(x)|
\]

\[
\leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| |z_j^*(E_jx)|
\]

where \( E_j \) is the smallest interval containing the support of \( z_j^* \) intersected with the support of \( x \). Continuing the above calculation we have
CASE 3: Let \( z^* \) be given by (33). We can of course assume \( z^*(x) \neq 0 \) thus let \( i_0 \) be the smallest natural number such that
\[
\bigcup_{i=1}^{i_0} \text{supp}(z^*_i) \cap \{n_0 + 1, n_0 + 2, \ldots\} \neq \emptyset.
\]
Then by the definition of \( n_0 \) we have that \( \sigma(\beta z_1^*, \beta z_2^*, \ldots, \beta z_i^*) > j_0 \) for \( i_0 + 1 \leq i \leq \ell \). Thus
\[
(37)
\]
\[
|z^*(x)| \leq \frac{1}{\sqrt{f(\ell)}} \left( \frac{1}{\sqrt{f(m_{i_0})}} \left| \sum_{i=1}^{m_{i_0}} \alpha_{i_0,j} z^*_{i_0,j}(x) \right| + \frac{1}{\sqrt{f(\ell)}} \frac{1}{\sqrt{f(m_{i_0+1})}} \sum_{j=1}^{m_{i_0+1}} |\alpha_{i_0+1,j}| |z^*_{i_0+1,j}(x)| \right) + \cdots
\]
\[
\leq \frac{1}{\sqrt{f(\ell)}} \|y\|_{s_2} + \varepsilon + \frac{1}{\sqrt{f(\ell)}} \frac{1}{\sqrt{f(m_{i_0+1})}} \sum_{1 \leq j \leq m_{i_0+1}: z^*_{i_0+1,j} \neq 0} |\alpha_{i_0+1,j}| |z^*_{i_0+1,j}(x)| \right) + \cdots \quad \text{(by CASE 2)}
\]
\[
\leq \frac{1}{\sqrt{f(\ell)}} \|y\|_{s_2} + \varepsilon + \frac{1}{\sqrt{f(\ell)}} \frac{1}{\sqrt{f(j_0)}} N(\|y\|_{s_2} + \varepsilon),
\]
where the last inequality is valid since \( |\alpha_{i,j}| \leq 1, |z^*_{i,j}(x)| \leq \|z^*_{i,j}\|_n \leq \|x\|_n \leq \|y\|_{s_2} + \varepsilon \), and \( m_i \geq j_0 \) for all \( i_0 + 1 \leq i \leq \ell \), (by the choice of \( n_0 \)), and there are at most \( N \) many indices \( (i,j) \) for which \( z^*_{i,j}(x) \neq 0 \). The last expression in equation (37) is at most equal to \( \|y\|_{s_2} + 3\varepsilon \) which finishes the proof.

For the following remark we will need a bit more notation. If \( X \) is a Banach space having an unconditional basis \( (e_n) \) and \( X_2 \) is the 2-convexification of \( X \) then for \( x = \sum a_n e_n \in X \) then there is a canonical image of \( x \) in \( X_2 \) which we define as \( \sqrt{x} = \sum \text{sign}(a_n) \sqrt{|a_n|} \sqrt{e_n} \in X_2 \) (where \( \sqrt{e_n} \) denotes the basis of \( X_2 \)).

**Remark 4.2.** Let \( 1 \leq p < \infty \). Let \( X \) be a Banach space with an unconditional basis \( (e_i) \) and let \( (\sqrt{e_i}) \) be the basis of \( X_2 \). Then \( p \) is in the Krivine set of \( (e_i) \) if and only if \( 2p \) is in the Krivine set of \( (\sqrt{e_i}) \).

**Proof.** Let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) be given. Since \( p \) is in the Krivine set of \( X \), there exists a block sequence \( (v_i)_{i=1}^n \) of \( (e_i) \) such that for any scalars \( (a_i)_{i=1}^n \) we have that
\[
\frac{1}{1 + \varepsilon} \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \leq \| \sum_{i=1}^{n} a_i v_i \| \leq (1 + \varepsilon) \left( \sum_{i=1}^{n} a_i^p \right)^{1/p}
\]

Let \( w_i = \sqrt{v_i} \in X_2 \), and scalars \((a_i)_{i=1}^{n}\) then

\[
\| \sum_{i=1}^{n} a_i w_i \|_{X_2} = \| \sum_{i=1}^{n} a_i^2 v_i \|_{X}^{1/2} \leq (1 + \varepsilon) \left( \sum_{i=1}^{n} a_i^{2p} \right)^{1/2p}
\]

and

\[
\| \sum_{i=1}^{n} a_i w_i \|_{X_2} = \| \sum_{i=1}^{n} a_i^2 v_i \|_{X}^{1/2} \geq \frac{1}{1 + \varepsilon} \left( \sum_{i=1}^{n} a_i^{2p} \right)^{1/2p}.
\]

The proof of the converse is similar. \(\square\)

It is known [19] that the Krivine set of the unit vector basis of \( S \) consists of the singleton \( \{1\} \). Thus by Remark 4.2 we have:

**Remark 4.3.** The Krivine set of the unit vector basis of \( S_2 \) consists of the singleton \( \{2\} \).

**Proposition 4.4.** Let \( X, Y \) be Banach spaces with unconditional bases, \((x_n)\) be a basic sequence in \( X \), \((y_n)\) be a basic sequence in \( Y \) such that \((x_n) >> (y_n)\). Then \((\sqrt{x_n}) >> (\sqrt{y_n})\) where \((\sqrt{x_n})\) and \((\sqrt{y_n})\) are the canonical images of \((x_n)\) and \((y_n)\) in \( X_2 \) and \( Y_2 \) respectively.

**Proof.** Note that

\[
\lim_{n \to \infty} \inf_{A \subseteq \mathbb{N} : |A| = n} \left\{ \sum_{i \in A} \sqrt{x_i} \right\}_{X_2} = \lim_{n \to \infty} \inf_{A \subseteq \mathbb{N} : |A| = n} \left\{ \sum_{i \in A} x_i \right\}_{X}^{1/2} = \infty.
\]

Also

\[
\Delta_{(\sqrt{x_n}), (\sqrt{y_n})}(\varepsilon) = \sup \left\{ \| \sum_{i} a_i \sqrt{y_i} \|_{Y_2} : |a_i| \leq \varepsilon \text{ and } \left\| \sum_{i} a_i \sqrt{x_i} \right\|_{X_2} = 1 \right\}
\]

\[
= (\sup \left\{ \| \sum_{i} a_i^2 y_i \|_Y : |a_i|^2 \leq \varepsilon^2 \text{ and } \left\| \sum_{i} a_i^2 x_i \right\|_X = 1 \right\})^{1/2}
\]

\[
\leq (\Delta_{(x_n), (y_n)}(\varepsilon^2))^{1/2}.
\]

The result follows immediately from (38) and (39). \(\square\)

It has been shown in [2, Proposition 2.1] that if \((e_n)\) is the unit vector basis of \( \ell_1 \) and \((f_n)\) is a normalized subsymmetric basic sequence which is not equivalent to \((e_n)\) then \((e_n) >> (f_n)\). Thus since the unit vector basis of \( S \) is normalized and subsymmetric we have that the unit vector basis of \( \ell_1 \) s.c. dominates the unit vector basis of \( S \). Thus Proposition 4.4 gives:

**Remark 4.5.** The unit vector basis of \( \ell_2 \) s.c. dominates the unit vector basis of \( S_2 \).

**Theorem 4.6.** There exists an infinite dimensional subspace \( Y \) of \( D \) having a basis and \( T \in \mathcal{L}(Y, D) \) such that \( T \notin \mathcal{C}_i Y \rightarrow D + \mathcal{K}(Y, D) \).

**Proof.** We will refer to \((s_i)\) as the unit vector basis of the Schlumprecht space \( S \) and \((\sqrt{s_i})\) as the unit vector basis of \( S_2 \). Then apply Theorem 0.8 for \( p = 2 \) and the spreading model for the unit vector basis of \( D \). By Proposition 4.1 we have that \((\sqrt{s_i})\) is the spreading model of the unit vector basis of \( D \). By Remark 4.3 we have that \( 2 \) is in the Krivine set of \((\sqrt{s_i})\). By Remark 4.5 we have that the unit vector basis of \( \ell_2 \) s.c. dominates \((\sqrt{s_i})\). Thus by
Theorem 0.8 we have there exists an infinite dimensional subspace $Y$ of $D$ and $T \in \mathcal{L}(Y, D)$ such that $T \not\in \mathcal{C}_{iY 	o D} + \mathcal{K}(Y, D)$.

\begin{thebibliography}{99}


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