On Banach Spaces containing $l_p$ or $c_0$

George Androulakis  
Nigel Kalton*  
Adi Tcaciuc†

February 20, 2009

Abstract

We use the Gowers block Ramsey theorem to characterize Banach spaces containing isomorphs of $\ell_p$ (for some $1 \leq p < \infty$) or $c_0$.

1 Introduction

A result of Zippin [Z] gives a characterization of the unit vector basis of $c_0$ and $l_p$. He showed that a normalized basis of a Banach space such that all normalized block bases are equivalent, must be equivalent to the unit vector basis of $c_0$ or $l_p$ for some $1 \leq p < \infty$. Let $1 \leq p \leq \infty$ A Banach space $X$ with a basis $(x_i)_i$ is called asymptotic-$l_p$ (asymptotic-$c_0$ if $p = \infty$) [M-TJ] if there exists $K > 0$ and an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n$, if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_{i=f(n)}^\infty$, then $(y_i)_{i=1}^n$ is equivalent to the unit vector basis of $l_p^n$. In [F-F-K-R] Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski gave necessary and sufficient conditions for finding asymptotic-$l_p$ subspaces, for a fixed $1 \leq p \leq \infty$, in an arbitrary Banach space. More precisely, they proved

**Theorem** (FFKR). Let $p \geq 1$ and let $X$ be a Banach space with the following property:

For any infinite dimensional subspace $Y \subseteq X$ there exists a constant $M_Y$ such that for any $n$ there exist infinite dimensional subspaces $U_1, U_2, \ldots, U_n$ of $Y$ with the

*This author was supported by NSF grant DMS-0555670
†This author was supported by NSERC

2000 Mathematics Subject Classification Primary: 46B20, Secondary: 46B40, 46B03
property that any normalized sequence \((u_1, u_2, \ldots, u_n)\) with \(u_i \in U_i\) for any \(i \leq n\), is 
MY-equivalent to the unit vector basis of \(l_p^n\).

Then \(X\) contains an asymptotic-\(l_p\) subspace.

In [Tc] we consider similar decompositions for which any two \(n\)-tuples as above are uniformly equivalent to each other (with the equivalence constant independent of \(n\)) and obtain the existence of asymptotic-\(l_p\) subspaces. Our current results are in the same direction. In Theorem 4 we show that if a Banach space has the property that every closed subspace contains two sequences of infinite-dimensional closed subspaces which are comparable (see the formal definition below) then it contains a copy of \(\ell_p\) for some \(1 \leq p < \infty\) or \(c_0\). This may be regarded as a characterization of spaces containing \(l_p\) or \(c_0\). In Theorem 6 we show that if a Banach space \(X\) is saturated with sequences of infinite dimensional subspaces \(E_n, n \in \mathbb{N}\), such that all normalized sequences \((x_n)\) with \(x_n \in E_n\) are tail equivalent, then \(X\) must contain a subspace isomorphic to \(\ell_p\) or \(c_0\). For the proofs we make essential use of Gowers’ block Ramsey theorem [G].

Acknowledgments. The third named author wishes to thank Professor Nicole Tomczak-Jaegermann for many useful discussions.

2 The main results

We first recall the Gowers block Ramsey theorem. Let \(X\) be a Banach space with a basis \((e_i)_i\) and let \(\Sigma\) be the subset of all finite normalized block sequences of \((e_i)_i\).

Given a set \(\sigma \in \Sigma\) we consider the following two players game (the Gowers game). S (the subspace player) chooses a block subspace \(X_1\) of \(X\) and V (the vector player) chooses a normalized vector \(x_1 \in X_1\). Then S chooses a block subspace \(X_2\) and V chooses \(x_2 \in X_2\). The play alternates S,V,S,V,... Player V wins the game if at some point the sequence \((x_1, x_2, \ldots, x_n)\) belongs to \(\sigma\). We say that V has a winning strategy if no matter what sequence of subspaces S chooses, V can win the game.

If \(Y\) is a block subspace of \(X\) we say that \(\sigma\) is large for \(Y\) if every block subspace of \(Y\) contains an element of \(\sigma\). We say that \(\sigma\) is strategically large for \(Y\) if player V has a winning strategy for the above game when S is restricted to subspaces of \(Y\).

Let \(\Delta = (\delta_1, \delta_2, \ldots)\) be a sequence of positive numbers. The set \(\sigma_\Delta\), the \(\Delta\)-enlargement of \(\sigma\), will stand for the set of all sequences \((x_1, x_2, \ldots, x_n)\) \(\in \Sigma\) such that there exists a sequence \((y_1, \ldots, y_n)\) \(\in \sigma\) with \(\|x_i - y_i\| < \delta_i\), for all \(i \leq n\).

**Theorem 1.** (Gowers’s block Ramsey Theorem) Let \(\sigma \in \Sigma\) be large for \(X\) and let \(\Delta\)
be a sequence of positive numbers. Then there exists a block subspace $Y$ of $X$ such that $\sigma_\Delta$ is strategically large for $Y$.

The following lemma is an application of Gowers’s block Ramsey Theorem.

**Lemma 2.** Let $X$ be a Banach space with a basis. Then there exists $1 \leq p \leq \infty$ and an infinite dimensional block subspace $Y$ of $X$ such that for every sequence $(Y_i)_{i=1}^\infty$ of infinite dimensional block subspaces of $Y$ and for all $n$ there exists a normalized block sequence $(y_i)_{i=1}^n$ in $Y$ with $y_i \in Y_i$ for $1 \leq i \leq n$ and $(y_i)_{i=1}^n$ is $2-$ equivalent to the unit vector basis of $l^n_p$.

For the proof of this lemma we will need to define the stabilized Krivine set of a Banach space. If $X$ is a Banach space with a basis and $W$ is an infinite dimensional block subspace of $X$, let $K(W)$ be the set of $p$’s in $[1, \infty]$ such that $l_p$ is finitely block represented in $W$, ($p = \infty$ if $c_0$ is finitely block represented in $W$). Then $K(W)$ is a closed non-empty subset of $[1, \infty]$ [K]. Note that if $W_2$ is an infinite dimensional block subspace of $W_1$ then $K(W_2) \subseteq K(W_1)$. Moreover, if $W_2$ has finite codimension in $W_1$ then $K(W_2) = K(W_1)$. Using these properties it is easy to show that there exists an infinite dimensional block subspace $W$ of $X$ such that $K(W) = K(V)$ for all infinite dimensional block subspaces $V$ of $W$, (else a transfinite induction gives a contradiction). The set $K(W)$ is called a stabilized Krivine set for $X$.

**Proof.** Let $W$ be an infinite dimensional block subspace of $X$ such that $K(W)$ is a stabilized Krivine set for $X$. Fix $p \in K(W)$ and for any $n \in \mathbb{N}$ let $\sigma_n$ be the set of all finite normalized block sequences of $W$ of length $n$ such that they are $2$-equivalent to the unit vector basis of $l^n_p$. The conclusion of the Lemma follows easily if we can find a block subspace $Y$ such that, for any $n$, $\sigma_n$ is strategically large for $Y$.

Note that for any $n$, $\sigma_n$ is large in any infinite dimensional block subspace $V$ of $W$. By applying Theorem 1 repeatedly, we obtain a nested sequence $V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_n \supset \cdots$ of block subspaces of $W$ such that, for any $n$, $\sigma_n$ is strategically large for $V_n$. Note that we do not enlarge $\sigma_n$ since we can replace the 2 in “2-equivalent” by $1 + \varepsilon$ for any $\varepsilon > 0$.

Let $Y$ be a diagonal block subspace, that is a subspace generated by a block basis $v_1, v_2, \ldots$ with $v_n \in V_n$ for every $n$. We claim that $\sigma_n$ is strategically large for $Y$, for any value of $n$. Indeed fix $n \in \mathbb{N}$ and denote by $[Y]_n$ the $n$-tail of $Y$, that is the subspace generated by $(v_j)_{j \geq n}$. Note that $[Y]_n \subseteq V_n$. Consider a typical Gowers game in $Y$. For any choice of a block subspace $Z$ of $Y$ the subspace player makes, the vector player chooses a vector $z \in Z \cap [Y]_n$ as if the game was played inside $V_n$ and the subspace player picked $Z \cap [Y]_n$. Since the vector player has a winning
strategy for the game played in $V_n$, it follows that after finitely many steps the finite block sequence he chooses belongs to $\sigma_n$. Therefore the vector player has a winning strategy for the game played in $Y$ as well. This proves that $\sigma_n$ is strategically large for $Y$, which finishes the proof of the lemma.

Let $E = (E_j)_{j=1}^\infty$ be a sequence of nonzero subspaces of a Banach space $X$ and let $F = (F_j)_{j=1}^\infty$ be a sequence of nonzero subspaces in a Banach space $Y$. We will say that $E$ is $C$-dominated by $F$, and we write $E \prec_C F$ if for any $n$ we have

$$\| \sum_{j=1}^n x_j \| \leq C \| \sum_{j=1}^n y_j \|$$

whenever $x_j \in E_j$, $y_j \in F_j$ with $\| x_j \| = \| y_j \|$ for $1 \leq j \leq n$.

The two sequences $E$ and $F$ are called $C$-comparable if $E \prec_C F$ or $F \prec_C E$. $E$ and $F$ are $C$-equivalent if there exist constants $C_1$ and $C_2$ with $C_1 C_2 = C$ such that $E \prec_C F$ and $F \prec_C E$.

Notice that the sequence $E$ is comparable to itself if and only if it is equivalent to itself and this is in turn equivalent to the fact that $E = (E_j)_{j=1}^\infty$ is an absolute Schauder decomposition of its closed linear span $[E]$.

Note that $E$ is $C$-dominated by the canonical one-dimensional decomposition of $\ell_p$ (or $c_0$ when $p = \infty$) if and only if $E$ satisfies an upper $\ell_p$-estimate with constant $C$, i.e.:

$$\| \sum_{j=1}^n x_j \| \leq C \left( \sum_{j=1}^n \| x_j \|^p \right)^{1/p}, \quad x_j \in E_j, \; n = 1, 2, \ldots$$

or

$$\| \sum_{j=1}^n x_j \| \leq C \max_{1 \leq j \leq n} \| x_j \|, \quad x_j \in E_j, \; n = 1, 2, \ldots$$

when $p = \infty$. Similarly $E$ $C$-dominates the canonical one-dimensional decomposition of $\ell_p$ if and only if $E$ satisfies a lower $\ell_p$-estimate with constant $C$.

Recall that a basic sequence $(x_n)_{n=1}^\infty$ has a block upper (respectively, block lower) $p$-estimate if there is a constant $C$ so that for all block basic sequences $(u_n)_{n=1}^\infty$ the sequence $([u_n])_{n=1}^\infty$ has an upper (respectively, lower) $\ell_p$-estimate with constant $C$.

**Theorem 3.** Let $X$ be a separable Banach space with a basis $(e_j)_{j=1}^\infty$ and suppose $1 \leq p \leq \infty$.
(i) Assume that every closed subspace of $X$ contains a sequence $E$ of infinite-dimensional subspaces with an upper $\ell_p$ (or $c_0$ when $p = \infty$)-estimate. Then $(e_j)_{j=1}^\infty$ has a block basic sequence with an block upper $p$-estimate.

(ii) Assume that every closed subspace of $X$ contains a sequence $E$ of infinite-dimensional subspaces with a lower $\ell_p$ (or $c_0$ when $p = \infty$)-estimate. Then $(e_j)_{j=1}^\infty$ has a block basic sequence with a block lower $p$-estimate.

**Proof.** We prove only (i) as (ii) is similar. We may assume that every block subspace contains a sequence $E$ of block subspaces with an upper $\ell_p$-estimate. For each block subspace $W$ let $C(W)$ denote the infimum of all constants $C$ so that $W$ contains a sequence $E$ of block subspaces with an upper $\ell_p$-estimate with constant $C$. We claim that there exists an infinite dimensional block subspace $Y$ of $X$ and a constant $C < \infty$ such that for each infinite dimensional block subspace $Z$ of $Y$ we have that $C(Z) < C$. Indeed, otherwise there exists a decreasing sequence of block subspaces $Z_n$ of $X$ such that $C(Z_n) > n$. If we choose a sequence $(w_j)_{j=1}^\infty$ of successive, linearly independent block vectors with $w_j \in Z_j$ for each $j$, then the infinite dimensional block subspace $W$ spanned by $(w_j)_{j=1}^\infty$ contains a sequence of infinite-dimensional subspaces $E = (E_j)_{j=1}^\infty$ with an $\ell_p$-upper estimate with constant $C$, say. Picking $n > C$ and considering the sequence $(E_j \cap Z_n)_{j=1}^\infty$, we have a contradiction. This contradiction proves the above claim.

Thus we may assume that for original basis we have the property that $C(W) < C$ for every block subspace. Now define the set $\sigma$ to consist of all normalized finite block basic sequences $(u_1, \ldots, u_n)$ so that for some $a_1, \ldots, a_n$ with $|a_1|^p + \cdots + |a_n|^p = 1$

$$\|a_1u_1 + \cdots + a_nu_n\| > C + 2.$$

If the conclusion of the Theorem is false, this set is large. Let $\Delta = (2^{-i})_{i=1}^\infty$, then by Theorem 1 $\sigma_\Delta$ is strategically large for some block subspace $Y$. Let $E = (E_j)_{j=1}^\infty$ be a sequence of infinite-dimensional block subspaces of $Y$ with an upper $\ell_p$-estimate with constant at most $C$. If the subspace player $S$ uses the strategy $E$ then the vector player $V$ may select normalized vectors $v_j \in E_j$ so that for some $n$ there exists $(u_1, \ldots, u_n) \in \sigma$ so that $\|u_j - v_j\| < 2^{-j}$ for $j = 1, 2, \ldots, n$. Now for an appropriate $a_1, \ldots, a_n$ with $|a_1|^p + \cdots + |a_n|^p = 1$ we have

$$\|\sum_{j=1}^n a_jv_j\| \geq \|\sum_{j=1}^n a_ju_j\| - 1 \geq C + 1$$

which gives a contradiction. This proves (i). □
Theorem 4. Let $X$ be a separable Banach space with the property that every infinite-dimensional closed subspace contains two comparable sequences $E$ and $F$ of infinite-dimensional closed subspaces. Then $X$ contains a copy of $\ell_p$ for some $1 \leq p < \infty$ or $c_0$.

Proof. We first assume that $X$ has a basis and then by Lemma 2, we may pass to a block subspace $Y$ so that for a suitable $p$, whenever $E = (E_j)_{j=1}^\infty$ is a sequence of infinite-dimensional subspaces of $Y$ then for any $n \in \mathbb{N}$ there exist $y_j \in E_j$ for $j = 1, 2, \ldots, n$ so that $(y_j)_{j=1}^n$ is 2-equivalent to the canonical $\ell_p^n$-basis.

By our assumption, for any closed infinite dimensional subspace $Z$ of $Y$ let $E$ and $F$ be two comparable sequences of infinite dimensional subspaces of $Z$, say $E = (E_j)$ is dominated by $F = (F_j)$. By Lemma 2 for any $n \in \mathbb{N}$ there exists $y_j \in E_j$ $(j = 1, \ldots, n)$, such that $(y_j)_{j=1}^n$ is 2-equivalent to the unit vector basis of $\ell_p^n$. Thus $F$ has a block lower $p$ estimate. Similarly, $E$ has a block upper $p$-estimate. Applying Theorem 3 (i) and (ii) one after another we can pass to a block basic sequence which has both a block upper and a block lower $p$-estimate, i.e. is equivalent to the canonical basis of $\ell_p$.

Corollary 5. Let $X$ be a separable Banach space with the property that every infinite-dimensional closed subspace contains a subspace with an absolute Schauder decomposition of infinite-dimensional subspaces. Then $X$ contains a copy of $\ell_p$ for some $1 \leq p < \infty$ or $c_0$.

We remark that this Corollary could easily be deduced from the result in [Tc].

A sequence $(x_n)_n$ will be called C-tail equivalent if for any $N \in \mathbb{N}$, there exists $k > N$ such that $(x_n)_{n=1}^\infty$ is C-equivalent to $(x_n)_{n=k}^\infty$. In particular, a subsymmetric basic sequence is tail equivalent.

Theorem 6. Let $X$ be a Banach space having the property that for any infinite dimensional closed subspace $Y$ of $X$ there exist a constant $C_Y$ and infinite dimensional subspaces $E = (E_i)_{i=1}^\infty$ of $Y$ such that all normalized sequences $(x_n)_n$ with $x_n \in E_n$ for all $n$ are $C_Y$-tail equivalent. Then $X$ contains a basic sequence equivalent to the unit vector basis of $\ell_p$ or $c_0$.

Proof. By passing to a subspace and relabeling assume that $X$ has a basis and $M$ is its basis constant. According to Lemma 2 let $Y$ be a block subspace of $X$ such that for every sequence $(Y_i)_{i=1}^\infty$ of infinite dimensional block subspaces of $Y$ and for all $n \in \mathbb{N}$ there exists a normalized sequence $(y_i)_{i=1}^n$ with $y_i \in Y_i$ for $1 \leq i \leq n$ and $(y_i)_{i=1}^n$ is 2-equivalent to the unit vector basis of $\ell_p^n$. 

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By our hypothesis there exists a sequence $E = (E_j)$ of infinite dimensional subspaces of $Y$ such that all normalized sequences $(x_n)$ with $x_n \in E_n$ are $C$-tail equivalent. Using a standard standard perturbation arguments we can assume that $E$ consists of block subspaces. Applying Lemma 2 for $(E_i)_{i=1}^\infty$ and for $n = 2$ we obtain normalized block vectors $z_1 \in E_1$ and $z_2 \in E_2$ such that $\{z_1, z_2\}$ is 2-equivalent to the unit vector basis of $l^2_p$. Next we apply again Lemma 2 for $(E_i)_{i=3}^\infty$ and for $n = 3$ and obtain normalized block vectors $z_3 \in E_3$, $z_4 \in E_4$ and $z_5 \in E_5$ such that $\{z_3, z_4, z_5\}$ is 2-equivalent to the unit vector basis of $l^3_p$. We continue in this manner; thus we built inductively a normalized block sequence $(z_i)_i$ with $z_i \in E_i$ for all $i$ such that $\{z_1, z_2\}$ is 2-equivalent to the unit vector basis of $l^2_p$, $\{z_3, z_4, z_5\}$ is 2-equivalent to the unit vector basis of $l^3_p$, $\{z_6, z_7, z_8, z_9\}$ is 2-equivalent to the unit vector basis of $l^4_p$ and so on. Clearly $(z_i)_i$ is a block basic sequence with basis constant at most $M$.

Fix $n \in \mathbb{N}$. From the definition of tail equivalence we can find $N$ large enough such that $(z_i)_{i=1}^\infty$ is $C$-equivalent to $(z_i)_{i=N+1}^\infty$ and $\{z_{N+1}, z_{N+2}, \ldots, z_{N+n}\}$ overlaps with at most two finite sequences of $z$’s as above. In other words, there exists $k \leq n$ such that $(z_i)_{i=N+1}^{N+k}$ is 2-equivalent to the unit vector basis of $l^k_p$ and $(z_i)_{i=N+k+1}^{N+n}$ is 2-equivalent to the unit vector basis of $l^{n-k}_p$. Then it easily follows that $\{z_{N+1}, z_{N+2}, \ldots, z_{N+n}\}$ is $16(M + 1)$-equivalent to the unit vector basis of $l^n_p$. Since $(z_i)_{i=1}^\infty$ is $C$-equivalent to $(z_i)_{i=N+1}^\infty$, it follows that $(z_i)_{i=1}^n$ is $16C(M + 1)$-equivalent to the unit vector basis of $l^n_p$, and this finishes the proof. ■

References


Department of Mathematics, University of South Carolina, Columbia, SC 29208, United States  
giorgis@math.sc.edu

Department of Mathematics, University of Missouri, Columbia, MO 65211, United States  
nigel@math.missouri.edu

Mathematics and Statistics Department, Grant MacEwan College, Edmonton, Alberta, Canada T5J P2P, tcaciuc@math.ualberta.ca