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Generators of quantum Markov semigroups

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Quantum Markov Semigroups (QMSs) originally arose in the study of the evolutions of irreversible open quantum systems. Mathematically, they are a generalization of classical Markov semigroups where the underlying function space is replaced by a non-commutative operator algebra. In the case when the QMS is uniformly continuous, theorems due to the works of Lindblad [Commun. Math. Phys. 48, 119-130 (1976)], Stinespring [Proc. Am. Math. Soc. 6, 211-216 (1955)], and Kraus [Ann. Phys. 64, 311-335 (1970)] imply that the generator of the semigroup has the form $L(A) = \sum_{n=1}^{\infty} V_n^* A V_n + GA + AG^*$, where $V_n$ and $G$ are elements of the underlying operator algebra. In the present paper, we investigate the form of the generators of QMSs which are not necessarily uniformly continuous and act on the bounded operators of a Hilbert space. We prove that the generators of such semigroups have forms that reflect the results of Lindblad and Stinespring. We also make some progress towards forms reflecting Kraus’ result. Finally, we look at several examples to clarify our findings and verify that some of the unbounded operators we are using have dense domains. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4928936]

I. MOTIVATION AND OVERVIEW OF OUR RESULTS

In this section, we motivate and overview our results while precise definitions appear in Section II. In the 1970s, Ingarden and Kossakowski\textsuperscript{17,18} postulated that the time evolution of a statistically open system, as given by a one-parameter semigroup of linear operators acting on the trace-class operators of a separable Hilbert space $H$ statistically open system, in the Schrödinger picture, be given by a one-parameter semigroup of $B(H)$ acting on $\mathcal{B}(H)$, where each $T_t$ is positive and $\sigma$-weakly continuous, satisfying $T_t(1) = 1$ for all $t \geq 0$, and where the map $t \mapsto T_t A$ is $\sigma$-weakly continuous for each $A \in \mathcal{B}(H)$.

In 1976, Lindblad\textsuperscript{20} added to the formulation the condition that each $T_t$ be completely positive rather than simply positive, a condition which he justified physically. Results of Stinespring,\textsuperscript{25} Theorem 4 and Arveson,\textsuperscript{1} and Proposition 1.2.2 further justify this condition by proving that if an operator has a commutative domain or target space, then positivity and complete positivity are equivalent. Further, under the assumption that the map $t \mapsto T_t$ is uniformly continuous, the semigroup is called a uniformly continuous quantum Markov semigroup (QMS), the generator $L$ of the semigroup is bounded, and Lindblad was able to write $L$ in the form $L(A) = \phi(A) + G^* A + AG$, where $\phi$ is completely positive and $G \in \mathcal{B}(H)$. Using an earlier theorem of Stinespring,\textsuperscript{25} we can then write $\phi$ in the form $\phi(A) = \sum_{n=1}^{\infty} W_n^* A W_n$, where $W_n : \mathcal{K} \to \mathcal{H}$ is a bounded linear operator. When we combine the results of Stinespring and Kraus, we are then able to write $\phi$ in the form $\phi(A) = \sum_{n=1}^{\infty} V_n^* A V_n$, where $V_n \in \mathcal{B}(H)$. Lindblad’s original result was for QMSs on a hyperfinite
factor $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ (which includes the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$, see the work of Topping\textsuperscript{23}). A similar result to the work of Lindblad was given in that same year by Gorini, Kossakowski, and Sudarshan\textsuperscript{14} for QMSs on finite dimensional Hilbert spaces, and three years later, Christensen and Evans\textsuperscript{8} proved it for uniformly continuous QMSs on arbitrary von Neumann algebras. A nice exposition of these results is written by Fagnola.\textsuperscript{12} Another name for QMSs that appears in the literature is CP$^*$-semigroups, for example, see the work of Arveson.\textsuperscript{2} An important subclass of QMSs that has also attracted a lot of attention is the class of $E_0$-semigroups which was introduced by Powers.\textsuperscript{23}

While we study the question, “Given a QMS, what is the form of its generator and does it satisfy the Lindblad equation?” the converse question, “Given an operator on $\mathcal{B}(\mathcal{H})$ which has the Lindblad form, does there exist a QMS whose generator is the given operator?” has attracted a lot of attention in the literature. The study of this converse question was originated by Davies.\textsuperscript{9}

In this paper, we prove analogous results to Lindblad and Stinespring and make some progress towards proving Kraus’ result for the generator of a QMS acting on $\mathcal{B}(\mathcal{H})$ when we no longer assume that the semigroup is uniformly continuous. In this case, the generator $L$ is no longer bounded and so inevitably, much discussion on domains of operators and the density of such domains is required. Because of such difficulties, we introduce the notion of $U$-completely positive maps (for a linear subspace $U$ of $\mathcal{H}$) which is analogous to completely positive maps but is better suited for unbounded operators (see Definition 4.2). We are then able to show (see Theorem 4.3) that if $L$ denotes the generator of a QMS on $\mathcal{B}(\mathcal{H})$, then there exists a subspace $W$ of $\mathcal{H}$, a linear operator $K : W \to \mathcal{H}$, and a $W$-completely positive map $\phi : D(L) \to S(W)$ (where $D(L)$ denotes the domain of $L$ and $S(W)$ denotes the set of sesquilinear forms on $W \times W$) such that

$$\langle u, L(A)v \rangle = \phi(A)(u,v) + \langle Ku, Av \rangle + \langle u, AKv \rangle$$

for all $A \in D(L)$ and all $u, v \in W$. Unfortunately, this result does not tell us much about the subspace $W$ or the operator $K$. On the other hand, if we restrict ourselves to the domain algebra $\mathcal{A}$ of $L$, which is the largest $^*$-subalgebra of the domain of $L$ and was studied by Arveson,\textsuperscript{2} then we are able to find (see Theorem 4.5) an explicit subspace $U$ of $\mathcal{H}$ and a linear operator $G : U \to \mathcal{H}$ having an explicit formula and a $U$-completely positive map $\phi : \mathcal{A} \to S(U)$ such that

$$\langle u, L(A)v \rangle = \phi(A)(u,v) + \langle u, GAu \rangle + \langle GA^*u, v \rangle$$

for all $A \in \mathcal{A}$ and for all $u, v \in U$, where $\phi : \mathcal{A} \to S(U)$ is $U$-completely positive. With regard to Stinespring, we are able to show (see Theorem 4.7) that there exists a Hilbert space $\mathcal{K}$, a linear map $V : \mathcal{H} \to \mathcal{K}$, and a unital $^*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ so that $\phi(A)(u,w) = \langle Vu, \pi(A)Vw \rangle$ for all $u, w \in U$.

Theorems 4.5 and 4.7 are summarized in Corollary 4.8 which is the main result of our paper. Similar results have been given in the literature. In 1979, Davies\textsuperscript{10} showed for the generator of a strongly continuous semigroup of completely positive contractions on the space of trace class operators (the form of the predual semigroup of a QMS) to have a form similar to Lindblad’s form, it is enough for $(T_t)_{t \geq 0}$ to possess a pure stationary state, that is, an $x \in \mathcal{H}$ such that $T_t(|x\rangle\langle x|) = |x\rangle\langle x|$, for all $t \geq 0$, and for the dual semigroup to map compact operators to compact operators or for the dual semigroup to satisfy an extension property which is given in the work of Davies,\textsuperscript{10} p. 429. Our result differs from this result of Davies since we neither assume that the QMS is strongly continuous nor do we make the other above assumptions but rather we assume that a rank one operator is contained in the domain of the generator. Another result in this direction was given by Holevo\textsuperscript{16} in 1995 which is similar to ours but there are significant differences. First, Holevo assumes that there exists a dense subspace $D$ of $\mathcal{H}$ such that for any $x, y \in D$,

$$\lim_{t \to 0} \frac{1}{t} \langle x, T_t(A)y \rangle$$

exists for all $A \in \mathcal{B}(\mathcal{H})$. On the other hand, we do not need to assume that this limit exists for all $A \in \mathcal{B}(\mathcal{H})$. Instead, we work on Arveson’s domain algebra. Further, while we restrict the form of the sesquilinear form produced by the generator of a QMS to a subspace $U$ of $\mathcal{H}$ (just as Holevo restricted the domain of the form to $D$), we define a specific set $U$ and we obtain Lindblad’s form without needing to assume that $U$ is dense. In fact, in Section VI, we provide an example where $U$ is not dense and our theorem is still valid.
In Section V, we give partial results similar to the one given by Kraus but fall slightly short and discuss a possible way forward (see Proposition 5.6 and the discussion that follows it). Finally, in Section VI, we look at two examples to verify the form of their generators and to discuss their corresponding subspace \( U \) mentioned above. Example 6.2 is of particular interest as it gives a QMS whose generator is not in classical Lindblad form but is in the form we give in our main result.

An expanded version of the article appears in the second author’s Ph.D thesis which was prepared at the University of South Carolina under the supervision of the first author.

II. MATHEMATICAL BACKGROUND

In this section, we provide the necessary definitions and mathematical background that is needed for the rest of the paper. Throughout the paper, \( \mathcal{H} \) will denote a Hilbert space. To avoid confusion, we want to mention from the start that all of our inner products are linear in the second coordinate and conjugate linear in the first. Also, for \( x, y \in \mathcal{H} \), we define the rank one operator \(|x \rangle \langle y| : \mathcal{H} \to \mathcal{H} \) by \(|x \rangle \langle y| = \langle y, h \rangle x \). We will extensively use the \( \sigma \)-weak topology so it is worth recalling: on a general von Neumann algebra, the \( \sigma \)-weak topology is the \( w^\ast \) topology given by its predual (every von Neumann algebra has a predual). If the von Neumann algebra under consideration is \( B(\mathcal{H}) \), then the predual is given by the space of all trace class operators on \( \mathcal{H} \) which we will denote by \( L_1(\mathcal{H}) \). For a detailed description of the duality between \( B(\mathcal{H}) \) and \( L_1(\mathcal{H}) \), we refer the reader to the book of Pedersen.\(^{22,23}\) Theorem 3.4.13.

Definition 2.1. Let \( \mathcal{A} \) be a von Neumann algebra and let \( M_n \) be the set of all \( n \times n \) matrices with complex coefficients. Then, the algebraic tensor product \( \mathcal{A} \otimes M_n \) can be represented as the \( \sigma \)-algebra of \( n \times n \) matrices with entries in \( \mathcal{A} \). Every element \( A \in \mathcal{A} \otimes M_n \) can be written in the form

\[
A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij},
\]

where \( E_{ij} \) is the \( n \times n \) matrix with 1 in the \( (i,j) \)th position and zero everywhere else. If \( \mathcal{B} \) is also a von Neumann algebra and \( T : \mathcal{A} \to \mathcal{B} \) is a linear operator, then we define the linear map \( T^{(n)} : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n \) by

\[
T^{(n)} \left( \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij} \right) = \sum_{i,j=1}^{n} T(A_{ij}) \otimes E_{ij}.
\]

We say a map \( T : \mathcal{A} \to \mathcal{B} \) is positive if it maps positive elements to positive elements. It is called completely positive if \( T^{(n)} \) is positive for all \( n \in \mathbb{N} \). In the case that \( \mathcal{B} \) acts on a Hilbert space \( \mathcal{H} \), it can be proven that \( T \) is completely positive if

\[
\sum_{i,j=1}^{n} \langle h_i, T(A_{ij}^* A_j) h_j \rangle \geq 0
\]

for all \( n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{A} \) and \( h_1, \ldots, h_n \in \mathcal{H} \).\(^{12}\) Proposition 2.9.

Definition 2.2. Let \( \mathcal{A} \) be a von Neumann algebra. A Quantum Dynamical Semigroup (QDS) is a one-parameter family \( (T_t)_{t \geq 0} \) of \( \sigma \)-weakly continuous, completely positive, linear operators on \( \mathcal{A} \) such that

\begin{enumerate}
  \item \( T_0 = 1 \),
  \item \( T_{t+s} = T_t T_s \),
  \item for a fixed \( A \in \mathcal{A} \), the map \( t \mapsto T_t(A) \) is \( \sigma \)-weakly continuous.
\end{enumerate}

Further, if \( T_t(1) = 1 \) for all \( t \geq 0 \), then we say the quantum dynamical semigroup is Markovian or we simply refer to it as a QMS. If the map \( t \mapsto T_t \) is norm continuous, then we say the semigroup is uniformly continuous.

Note: If \( (T_t)_{t \geq 0} \) is a quantum Markov semigroup, then \( \|T_t\| = 1 \) for all \( t \geq 0 \). This is due to Dye and Russo.\(^{11}\) Corollary 1.
Let \( e \) be any fixed unit vector in a Hilbert space \( \mathcal{H} \). Then, if the QDS is \( \sigma \)-weakly dense, the generator \( L \) of the QDS is \( \sigma \)-weakly closed so if \( L \) has full domain, then it would be bounded. In this case, the QDS is then uniformly continuous (see the work of Hille and Phillips). It has been proven (see, for example, the work of Bratteli and Robinson, Proposition 3.1.6) that the domain of the generator \( L \) of a QDS is \( \sigma \)-weakly dense. However, if the QDS is not uniformly continuous, then generator \( L \) does not have full domain. Indeed, it is known (see the work of Bratteli and Robinson, Proposition 3.1.6) that \( L \) is \( \sigma \)-weakly closed so if \( L \) has full domain, then it would be bounded. In this case, the QDS is then uniformly continuous (see the work of Hille and Phillips).

**III. GENERATORS OF UNIFORMLY CONTINUOUS QUANTUM MARKOV SEMIGROUPS ON \( \mathcal{B}(\mathcal{H}) \)**

In this section, we recall some results for the form of the generator of a uniformly continuous QMS (which motivate our work on the consequent sections). As a motivation for Lindblad’s result, we start by describing a simple example of a QDS and its generator which comes from the work of Fagnola.

**Example 3.1.** Let \((U_t)_{t \geq 0}\) be a strongly continuous semigroup on a Hilbert space \( \mathcal{H} \). Then, define \( T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \), for all \( t \geq 0 \), by

\[
T_t(A) = U_t A U_t^*.
\]

Then, \((T_t)_{t \geq 0}\) is a quantum dynamical semigroup. Further, if \( G \) is the generator of \((U_t)_{t \geq 0}\) and \( G \) is bounded, then the generator, \( L \), of \((T_t)_{t \geq 0}\) is given by

\[
L(A) = GA + AG^*.
\]

This form should be compared with (1) of Theorem 3.2 which is due to the work of Lindblad. See the work of Parthasarathy, Proposition 30.14 where the specific form of \( G \) is proved or the work of Ziemke where other possible forms of \( G \) are given.

**Theorem 3.2 (Lindblad).** Let \( L \) be the generator of a uniformly continuous QMS on \( \mathcal{B}(\mathcal{H}) \). Let \( e \) be any fixed unit vector in \( \mathcal{H} \). If the operator \( G \) is defined on \( \mathcal{H} \) by

\[
G(x) = L(|x \rangle \langle e|) e - \frac{1}{2} \langle e, L(|e \rangle \langle e|) e \rangle x,
\]

then there exists a completely positive map \( \phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) such that

\[
L(A) = \phi(A) + GA + AG^*
\]

for all \( A \in \mathcal{B}(\mathcal{H}) \).

We have casually mentioned the results of Stinespring and Kraus earlier. Since we will attempt to generalize both, we feel it is necessary to give complete statements of them.
Theorem 3.3 (Stinespring). Let $\mathfrak{B}$ be a $C^*$-subalgebra of the algebra of all bounded operators on a Hilbert space $\mathcal{H}$ and let $\mathfrak{A}$ be a $C^*$-algebra with unit. A linear map $T : \mathfrak{A} \to \mathfrak{B}$ is completely positive if and only if it has the form
\[
T(A) = V^* \pi(A)V,
\]
where $(\pi, \mathcal{K})$ is a unital $^*$-representation of $\mathfrak{A}$ on some Hilbert space $\mathcal{K}$, and $V$ is a bounded operator from $\mathcal{H}$ to $\mathcal{K}$.

Theorem 3.4 (Kraus). Let $\mathfrak{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}$ be another Hilbert space. A linear map $T : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ is normal and completely positive if and only if it can be represented in the form
\[
T(A) = \sum_{j=1}^{\infty} V_j^* AV_j,
\]
where $(V_j)_{j=1}^{\infty}$ is a sequence of bounded operators from $\mathcal{K}$ to $\mathcal{H}$ such that the series $\sum_{j=1}^{\infty} V_j^* AV_j$ converge strongly.

IV. GENERATORS OF GENERAL QUANTUM MARKOV SEMIGROUPS ON $\mathcal{B}(\mathcal{H})$

In this section, we prove analogous expressions of (1) and (2) for the generator of a general QMS on $\mathcal{B}(\mathcal{H})$. The main result of the section as well as the main result of the paper is Corollary 4.8. Heading in this direction, we start with the following.

Theorem 4.1. Let $L$ be the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Then, there exists a family $(L_\epsilon)_{\epsilon > 0}$ of generators of uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$ such that
\[
L(A) = \lim_{\epsilon \to 0} L_\epsilon(A)
\]
for all $A \in D(L)$, where the limit is taken in the $\sigma$-weak topology. Thus, by Lindblad’s theorem, there exists a family $(\phi_\epsilon)_{\epsilon > 0}$ of normal completely positive operators on $\mathcal{B}(\mathcal{H})$ and a family $(G_\epsilon)_{\epsilon > 0}$ of bounded operators on $\mathcal{H}$ such that
\[
L_\epsilon(A) = \phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*
\]
for all $A \in \mathcal{B}(\mathcal{H})$.

Proof: Let $L$ be the generator for a quantum Markov semigroup $(U_t)_{t \geq 0}$. Let $L_\epsilon = L(1 - \epsilon L)^{-1}$. Then, for $\epsilon > 0$, $L_\epsilon$ is bounded and $\sigma$-weakly continuous, since by Bratteli and Robinson, Propositions 3.1.4 and 3.1.6, $(1 - \epsilon L)^{-1}$ is bounded and $\sigma$-weakly continuous and
\[
L(1 - \epsilon L)^{-1} = \frac{1}{\epsilon} (1 - (1 - \epsilon L)^{-1}).
\]
Define $U_{t, \epsilon} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $U_{t, \epsilon} = \exp(t L_\epsilon)$. Then, we know $(U_{t, \epsilon})_{t \geq 0}$ is a uniformly continuous semigroup. Further, we claim that $(U_{t, \epsilon})_{t \geq 0}$ is contractive. Indeed, by Bratteli and Robinson, Theorem 3.1.10, we have that $\|(1 - \epsilon L)^{-1}\| \leq 1$ for all $\epsilon > 0$, so
\[
\|U_{t, \epsilon}\| = \|e^{t L_\epsilon}\| \leq e^{-t/\epsilon} \sum_{n=0}^{\infty} \frac{(t/\epsilon)^n}{n!} \|(1 - \epsilon L)^{-1}\| \quad \text{by (4)}
\]
and so, $(U_{t, \epsilon})_{t \geq 0}$ is contractive. Further, since $L_\epsilon$ is $\sigma$-weakly continuous, we have, by Ref. 12, Proposition 3.9, that $U_{t, \epsilon}$ is $\sigma$-weakly continuous. Also, since $(U_t)_{t \geq 0}$ is Markovian, $1 \in D(L)$ and $L(1) = 0$, so
\[
L_\epsilon(1) = L(1 - \epsilon L)^{-1}(1) = (1 - \epsilon L)^{-1} L(1) = 0.
\]
Hence,

\[ U_{t,\epsilon}(1) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} L^n(1) = 1. \]

So, \( \|U_{t,\epsilon}\| = 1 \) and the norm is attained at 1 so, by Ref. 11, Corollary 1, \( U_{t,\epsilon} \) is positive. Now, \((U_t^{(n)})_{t \geq 0}\) is also a quantum Markov semigroup with generator \( L(n) \) so, following the above with \( U_t^{(n)} \) in place of \( U_t \) and \( L(n) \) in place of \( L \), we get that \( \exp(tL((1-\epsilon L(n)))^{-1}) \geq 0 \) for all \( n \in \mathbb{N} \). We now claim that \( L^{(n)(1-\epsilon L(n))^{-1}} = (L(1-\epsilon L)^{-1})^{(n)} \) which will prove that \( U_{t,\epsilon} \) is completely positive, since \((L(1-\epsilon L)^{-1})^{(n)} \) is the generator of the semigroup \((U_t^{(n)})_{t \geq 0}\). Indeed, for \( [A_{i,j}]_{i,j=1,\ldots,n} \in D(L) \otimes M_n(\mathbb{C}) \),

\[ (1-\epsilon L(n))(A_{i,j},\ldots,n) = (1-\epsilon L)(A_{i,j})_{i,j=1,\ldots,n}. \]

Hence,

\[ ((1-\epsilon L)^{-1}(1-\epsilon L))^{(n)}(A_{i,j},\ldots,n) = (1-\epsilon L)^{-1}(1-\epsilon L)(A_{i,j})_{i,j=1,\ldots,n} = [A_{i,j}]_{i,j=1,\ldots,n}, \]

which proves that \( (1-\epsilon L(n))^{-1} = ((1-\epsilon L)^{-1})^{(n)} \). Hence,

\[ L^{(n)(1-\epsilon L(n))^{-1}} = L^{(n)(1-\epsilon L)^{-1}} = (L(1-\epsilon L)^{-1})^{(n)}. \]

Therefore, \( U_{t,\epsilon} \) is completely positive for all \( t \geq 0 \) and \( \epsilon > 0 \). Then, by Theorem 3.2, there exists a completely positive map \( \phi_{\epsilon} \) and \( G_{\epsilon} \in \mathcal{B}(\mathcal{H}) \) such that

\[ L_{\epsilon}\phi(A) = \phi_{\epsilon}(A) + G_{\epsilon}A + AG_{\epsilon}^* \]

for all \( A \in \mathcal{B}(\mathcal{H}) \). Next, we claim that \( L_{\epsilon}(A) \rightarrow L(A) \) in the \( \sigma \)-weak topology for all \( A \in D(L) \). Let \( A \in \mathcal{B}(\mathcal{H}) \). First, we want to show \( (1-\epsilon L)^{-1}(A) \rightarrow A \) \( \sigma \)-weakly, so let \( \eta \) be an element of the predual \( L_1(\mathcal{H}) \) of \( \mathcal{B}(\mathcal{H}) \) and \( \gamma > 0 \). Since \( U_t(A) \rightarrow A \) \( \sigma \)-weakly, choose \( \delta > 0 \) so that for any \( t < \delta \), we have \( |\eta(U_t(A) - A)| < \gamma/2 \). Hence,

\[ \int_0^\delta e^{-t/\epsilon} |\eta(U_t(A) - A)| dt < \frac{\gamma}{2}. \]

Then,

\[ |\eta((1-\epsilon L)^{-1}(A)) - \eta(A)| = |\eta(e^{-1}(1-\epsilon) - L)^{-1}(A)) - \eta(A)| \]

\[ = \int_0^\infty e^{-t/\epsilon} |\eta(U_t(A)) dt - \eta(A)| \]

\[ \leq \int_0^\infty e^{-t/\epsilon} |\eta(U_t(A) - A)| dt + \int_0^\delta e^{-t/\epsilon} |\eta(U_t(A) - A)| dt \]

\[ \leq 2\|\eta\|\|A\| \int_0^\infty e^{-t/\epsilon} dt + \frac{\gamma}{2} \]

\[ = 2\|\eta\|\|A\|e^{-\delta/\epsilon} + \frac{\gamma}{2}. \]

So, pick \( \epsilon_0 > 0 \) so that for all \( 0 < \epsilon < \epsilon_0 \), we have \( e^{-\delta/\epsilon} < \gamma(4\|\eta\|\|A\|)^{-1} \). Then, we have that \( |\eta((1-\epsilon L)^{-1}(A)) - \eta(A)| < \gamma \) and therefore, \((1-\epsilon L)^{-1}(A) \rightarrow A \) \( \sigma \)-weakly for all \( A \in \mathcal{B}(\mathcal{H}) \). So, for \( A \in D(L) \), replace \( A \) with \( LA \) and we then have \( L(1-\epsilon L)^{-1}A \rightarrow LA \) \( \sigma \)-weakly since \( L(1-\epsilon L)^{-1}A = (1-\epsilon L)^{-1}LA \) for any \( A \in D(L) \). Hence, \( L_{\epsilon}(A) \rightarrow L(A) \) \( \sigma \)-weakly for all \( A \in D(L) \). Thus,

\[ \lim_{\epsilon \rightarrow 0} (\phi_{\epsilon}(A) + G_{\epsilon}A + AG_{\epsilon}^*) \]

which completes the proof. \( \Box \)

In Theorem 4.1, if \( A \in D(L^2) \), we actually get that \( L_{\epsilon}(A) \rightarrow L(A) \) in norm. Indeed, for \( A \in D(L) \),

\[ \|(1-\epsilon L)^{-1}A - A\| = \|(1-\epsilon L)^{-1} - (1-\epsilon L)^{-1}(1-\epsilon L)A\| = \epsilon \|(1-\epsilon L)^{-1}LA\| \leq \epsilon \|LA\| \]
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since \( \|(1 - \epsilon L)^{-1}\| \leq 1 \) for every \( \epsilon > 0 \) (see the work of Bratteli and Robinson, Proposition 3.1.10). So, for \( A \in D(L) \),
\[
\|(1 - \epsilon L)^{-1}A - A\| \leq \epsilon\|LA\| \to 0
\]
as \( \epsilon \to 0 \). Hence, if \( A \in D(L^2) \), then
\[
L_\epsilon(A) = L(1 - \epsilon L)^{-1}A = (1 - \epsilon L)^{-1}LA \to LA.
\]

For a general QMS on \( B(\mathcal{H}) \), we would not expect the completely positive part of the representation of the generator to be bounded. This leads us to the following definition.

**Definition 4.2.** Let \( U \) be a subspace of a Hilbert space \( \mathcal{H} \). A linear map \( \phi \) from a linear subspace \( \mathcal{R} \) of \( B(\mathcal{H}) \) to the set of sesquilinear forms on \( U \times U \) is **U-completely positive** if for any \( k \in \mathbb{N} \), any positive operator \( A = (A_{i,j})_{i,j=1,...,k} \in \mathcal{R} \otimes M_k(\mathbb{C}) \) and for all \( u_1, \ldots, u_k \in U \), we have that
\[
\sum_{i,j=1}^k \phi(A_{i,j})(u_i, u_j) \geq 0.
\]

We now proceed to give analogous forms to Lindblad’s form for the generator of a QMS.

**Theorem 4.3.** Let \( L \) be the generator of a QMS on the von Neumann algebra \( B(\mathcal{H}) \). Then, there exists a linear (not necessarily closed) subspace \( W \) of \( \mathcal{H} \), a \( W \)-completely positive map \( \phi \) from \( D(L) \) into the set of sesquilinear forms on \( W \times W \) and a linear operator \( K \) from \( W \) to \( \mathcal{H} \) such that
\[
\langle u, L(A)v \rangle = \phi(A)(u, v) + \langle Ku, Av \rangle + \langle u, AKv \rangle
\]
for all \( A \in D(L) \) and all \( u, v \in W \).

**Proof.** By Proposition 4.1, there exists a family \((\phi_\epsilon)_{\epsilon > 0}\) of normal completely positive operators on \( B(\mathcal{H}) \) and there exists a family \((G_\epsilon)_{\epsilon > 0} \subseteq B(\mathcal{H})\) such that
\[
L(A) = \lim_{\epsilon \to 0} (\phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*)
\]
for all \( A \in D(L) \), where the limit is taken in the \( \sigma \)-weak topology. Define \( W \subseteq \mathcal{H} \) by
\[
W = \{ u \in \mathcal{H} : \lim_{\epsilon \to 0} \langle x, G_\epsilon u \rangle \text{ exists for all } x \in \mathcal{H} \}.
\]
Then, define \( K \) on \( W \) by \( Ku = \text{weak-\( \lim_{\epsilon \to 0} G_\epsilon^*u \)}. \) Then, for \( A \in D(L), \)
\[
\langle u, L(A)v \rangle = \lim_{\epsilon \to 0} \langle u, (\phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*)v \rangle
\]
\[
= \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle + \langle Ku, Av \rangle + \langle u, AKv \rangle
\]
for all \( u, v \in W \). Further, since \( \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle \) exists for all \( A \in D(L) \) and for all \( u, v \in W \), define a linear map \( \phi \) from \( D(L) \) to the sesquilinear forms on \( W \times W \) by
\[
\phi(A)(u, v) = \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle.
\]
Let \( A = (A_{i,j})_{i,j=1,...,k} \in D(L) \otimes M_k(\mathbb{C}) \) be a positive operator and let \( u_1, \ldots, u_k \in W \). Since \( \phi_\epsilon \) is completely positive, we have that
\[
\sum_{i,j=1}^k \langle u_i, \phi_\epsilon(A_{i,j})u_j \rangle \geq 0.
\]
Since \( \langle u_i, \phi_\epsilon(A)v \rangle \to \phi(A)(u, v) \) for all \( A \in D(L) \) and \( u, v \in W \), we have that
\[
\sum_{i,j=1}^n \phi(A_{i,j})(u_i, u_j) \geq 0,
\]
which proves that \( \phi \) is \( W \)-completely positive. \( \Box \)
Theorem 4.5 gives a form of the generator similar to that of Theorem 4.3 with the added advantage that the subspace $W$ is replaced by a subspace $U$ which is easy to describe. The easy form of $U$ enables us to verify that it is dense in $\mathcal{H}$ in Example 6.1.

**Definition 4.4.** If $L$ is the generator of a QMS, then the **domain algebra** of $L$ is the largest $^*$-subalgebra of the domain of $L$, $D(L)$, and is shown by Arveson\cite{Arveson2} to be given by

$$ \mathcal{A} = \{ A \in D(L) : A^*A, AA^* \in D(L) \}. $$

**Theorem 4.5.** Let $L$ be the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Let $D(L)$ denote its domain and $\mathcal{A}$ denote its domain algebra. Assume there exists a unit vector $e \in \mathcal{H}$ such that $|e\rangle\langle e| \in D(L)$. Let $U$ be the linear subspace of $\mathcal{H}$ defined by $U = \{ x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A} \}$. Then, there exists a linear map $G : U \to \mathcal{H}$ and a $U$-completely positive map $\phi$ from $\mathcal{A}$ to the set of sesquilinear forms on $U \times U$ such that

$$ \langle u, L(A)v \rangle = \phi(A)(u,v) + \langle u, GA v \rangle + \langle GA^* u, v \rangle $$

for all $A \in \mathcal{A}$ and $u, v \in U$. Moreover, the linear map $G : U \to \mathcal{H}$ can be chosen to be

$$ G(u) = L(|u\rangle\langle e|)e - \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle u, \quad u \in U. $$

**Remark 4.6.** First, for the sake of clarity, we explain the definition of $U$. Note that by Definition 4.4, for $x \in \mathcal{H}, |x\rangle\langle e| \in \mathcal{A}$ is equivalent to having the following three conditions hold: $|x\rangle\langle e| \in D(L)$, $\langle |x\rangle\langle e| \rangle \circ |x\rangle\langle e| = \| x \|^2 |x\rangle\langle x| \in D(L)$, and $|x\rangle\langle e| \circ (|x\rangle\langle e|)^* = \| e \|^2 x \langle x| \in D(L)$. Thus, if $U$ contains non-zero vectors, then for some unit vector $e \in \mathcal{H}$, we have that $|e\rangle\langle e| \in D(L)$ and that is why this condition appears explicitly in the statement of Theorem 4.5.

**Proof of Theorem 3.9.** By Theorem 4.1, there exists a family $(L_e)_{e>0}$ of generators of uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$ such that $L(A) = \sigma$-weak-$\lim_{e \to 0} L_e(A)$ for every $A \in D(L)$. Also, there exist families of completely positive operators $(\phi_e)_{e>0}$ on $\mathcal{B}(\mathcal{H})$ and bounded operators $(G_e)_{e>0}$ on $\mathcal{H}$ such that

$$ L_e(A) = \phi_e(A) + G_eA + AG_e^* $$

for all $A \in D(L)$. Let $v \in U$ and let $A \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, we obtain $|Av\rangle\langle e| = A \circ |v\rangle\langle e| \in \mathcal{A}$. Then, using the explicit form for $G_e$ from Theorem 3.2, we have

$$ G_eAv = L_e(|Av\rangle\langle e|)e - \frac{1}{2} \langle e, L_e(|e\rangle\langle e|)e \rangle Av. \quad (5) $$

Since $|Av\rangle\langle e| \in \mathcal{A} \subseteq D(L)$, we obtain by Theorem 4.1 that $L_e(|Av\rangle\langle e|) \to L(|Av\rangle\langle e|)$ in the $\sigma$-weak topology. Thus, for any $u \in \mathcal{H}$, we obtain

$$ \langle u, L_e(|Av\rangle\langle e|)e \rangle \to \langle u, L(|Av\rangle\langle e|)e \rangle. \quad (6) $$

Also, by Theorem 4.1, since $T \in D(L)$, we have that $L_e(T) \to L(T)$ in the $\sigma$-weak topology and hence,

$$ \langle e, L_e(|e\rangle\langle e|)e \rangle \to \langle e, L(|\langle e\rangle\langle e|)e \rangle. \quad (7) $$

Thus, by (5)–(7), for any $u \in \mathcal{H}, v \in U$, and $A \in \mathcal{A}$, we have

$$ \langle u, G_eAv \rangle = \langle u, L_e(|Av\rangle\langle e|)e \rangle - \frac{1}{2} \langle e, L_e(|e\rangle\langle e|)e \rangle \langle u, Av \rangle \to \langle u, L(|Av\rangle\langle e|)e \rangle - \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle \langle u, Av \rangle = \langle u, GA v \rangle. $$

Similarly, for $u \in U, v \in \mathcal{H}$, and $A \in \mathcal{A}$, we have

$$ \langle u, AG_e^* v \rangle \to \langle GA^* u, v \rangle. $$
Thus, for \( u, v \in U \), and \( A \in \mathcal{A} \),
\[
\langle u, L(A)v \rangle = \lim_{\epsilon \to 0} \langle u, L_\epsilon(A)v \rangle = \lim_{\epsilon \to 0} \langle u, (\phi_\epsilon(A) + G_\epsilon + AG_\epsilon^*)v \rangle = \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle + \langle u, GAu \rangle + \langle GA^*u, v \rangle.
\]

Thus, \( \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle \) exists for all \( A \in \mathcal{A} \) and for all \( u, v \in U \), and therefore, define
\[
\phi(A)(u, v) = \lim_{\epsilon \to 0} \langle u, \phi_\epsilon(A)v \rangle.
\]

Let \( A = (A_{i,j})_{i,j=1,\ldots,k} \in \mathcal{A} \otimes \mathcal{M}_k(\mathbb{C}) \) be a positive operator and let \( u_1, \ldots, u_k \in U \). Since \( \phi_\epsilon \) is completely positive, we have that
\[
\sum_{i,j=1}^k \langle u_t, \phi_\epsilon(A_{i,j})u_t \rangle \geq 0.
\]

Since \( \langle u, \phi_\epsilon(A)v \rangle \to \phi(A)(u, v) \) for all \( A \in \mathcal{A} \) and \( u, v \in U \), we have that
\[
\sum_{i,j=1}^n \phi(A_{i,j})(u_t, u_j) \geq 0.
\]

Therefore, \( \phi \) is \( U \)-completely positive. \( \square \)

While restricting to the domain algebra helps us to understand the subspace \( U \) and the operator \( G \), it does come at a cost since the domain of the generator is \( \sigma \)-weakly dense, while there are examples of QMSs whose domain algebras are not very large. Indeed, Fagnola\(^{13}\) gives an example of a QMS on \( \mathcal{B}(L^2(0,\infty),\mathbb{C}) \), where \( \mathcal{A} \) is not \( \sigma \)-weakly dense in \( \mathcal{B}(L^2(0,\infty),\mathbb{C}) \). In Section VI, we will look at several examples where \( U \) is dense in \( \mathcal{H} \) and also verify the above form for the generator \( L \).

We will proceed by showing that we have analogous results to those of Stinespring’s work. In the next proposition, when we say a map \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \), where \( \mathcal{A} \) is a (not necessarily closed) unital \( ^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \), is a unital \( ^* \)-representation, we mean that it is a unital norm-continuous \( ^* \)-homomorphism.

**Theorem 4.7.** Suppose \( \mathcal{A} \) is a unital (not necessarily closed) \( ^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \), \( U \) is a (not necessarily closed) linear subspace of \( \mathcal{H} \), and \( \phi \) is a \( U \)-completely positive map from \( \mathcal{A} \) to the set of sesquilinear forms on \( U \times U \). Then, there exists a Hilbert space \( \mathcal{K} \), a unital \( ^* \)-representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) of norm equal to one, and a linear map \( V : U \to \mathcal{K} \) such that
\[
\phi(A)(u, w) = \langle Vu, \pi(A)Vw \rangle_{\mathcal{K}}
\]
for all \( u, w \in U \).

**Proof.** Define a sesquilinear form \( \langle \cdot, \cdot \rangle : (\mathcal{A} \otimes U) \times (\mathcal{A} \otimes U) \to \mathbb{C} \) by
\[
(x, y) = \sum_{i,j=1}^n \phi(A_i^*B_j)(u_t, v_j),
\]
where \( x = \sum_{i=1}^n A_i \otimes u_t \) and \( y = \sum_{j=1}^n B_j \otimes v_j \) (since we allow zero entries, we can have the same upper limit \( n \) in both sums). Since \( \phi \) is \( U \)-completely positive, \( (x, x) \geq 0 \) for all \( x \in \mathcal{A} \otimes U \), so \( (-, -) \) is a positive definite sesquilinear form. For \( x \in \mathcal{A} \otimes U \), let \( \|x\|_{(-, -)} = \sqrt{(x, x)} \). Let \( N = \{x \in \mathcal{A} \otimes U : (x, x) = 0\} \). Since \( (-, -) \) is a positive definite sesquilinear form, by the Cauchy-Schwartz inequality, \( N \) is a linear subspace of \( \mathcal{A} \otimes U \) and we have that the completion of \( (\mathcal{A} \otimes U) / N \), which we will denote by \( \mathcal{K} \), is a Hilbert space where the inner product is given by \( \langle x + N, y + N \rangle_{\mathcal{K}} = (x, y) \). Let \( \pi_0 : \mathcal{A} \to L(\mathcal{A} \otimes U) \) (where \( L(X) \) denotes the linear (not necessarily bounded) operators from \( X \) to \( X \)) defined by
\[
\pi_0(A)\left(\sum_{i=1}^n A_i \otimes u_t\right) = \sum_{i=1}^n AA_i \otimes u_t.
\]
Thus, in fact, so is linear. Then, for \( \parallel \cdot \parallel \) a norm equal to one. Indeed, let \( A \rightarrow B \) be a unital \( \ast \)-algebra, but we verify here that the representation has norm equal to one. Moreover, the linear map \( G \) is linear, \( \pi \) is linear, \( \pi(1_N) = 1_K \), and for \( A, B \in \mathcal{A} \), we have \( \pi(A) = \pi(A)^* \) and \( \pi(AB) = \pi(A)\pi(B) \) as in the proof of Stinespring. Follows immediately from Theorems 4.5 and 4.7. Proof. Further, let \( V : U \rightarrow K \), where \( Vu = 1 \otimes u + N \) for all \( u \in U \). Then, for \( u, w \in U \), and \( A \in \mathcal{A} \), we have that

\[
\langle Vu, \pi(A)Vw \rangle_K = \langle 1 \otimes u + N, A \otimes w + N \rangle_K = \langle 1 \otimes u, A \otimes w \rangle = \langle \pi(A)u, w \rangle.
\]

Any representation of a unital \( C^* \)-algebra into another is known to be bounded and in fact have norm equal to one (obtained at the identity, see the work of Sunder). The domain algebra \( \mathcal{A} \) is not closed so it is not a \( C^* \)-algebra, but we verify here that the representation \( \pi \) has norm equal to one. Indeed, let \( A \in \mathcal{A} \) and \( x + N \in K \). Then,

\[
\parallel \pi(A)(x + N) \parallel_K = \parallel \pi(A)x + N \parallel_K = \parallel \pi(A)x + N, \pi(A)x + N \parallel_K^{1/2} = \parallel \pi(A)x \parallel_K^{1/2}.
\]

Further, by (8),

\[
\parallel \pi(A)x \parallel_K^{1/2} \leq \parallel A^*A \parallel^{1/2}\parallel x \parallel_K^{1/2} = \parallel A \parallel \parallel x + N \parallel_K,
\]

and therefore, \( \parallel \pi(A) \parallel \leq \parallel A \parallel \) for all \( A \in \mathcal{A} \) and the proof is complete.

Theorems 4.5 and 4.7 bring us one step closer to the explicit form of the generator of a QMS. Our progress is summed up in the following which is the main result of our paper.

**Corollary 4.8.** Let \( L \) be the generator of a QMS on the von Neumann algebra \( \mathcal{B}(\mathcal{H}) \) and let \( \mathcal{A} \) be its domain algebra. Assume there exists a unit vector \( e \in \mathcal{H} \) such that \( |e \rangle \langle e | \in D(L) \). Let \( U \) be the linear subspace of \( \mathcal{H} \) defined by \( U = \{ x \in \mathcal{H} : \langle x | e \rangle \in \mathcal{A} \} \). Then, there exists a Hilbert space \( K \), a unital \( \ast \)-representation \( \pi : \mathcal{A} \rightarrow \mathcal{B}(K) \), and linear maps \( G : U \rightarrow \mathcal{H} \) and \( V : U \rightarrow K \) such that

\[
\langle u, L(A)w \rangle = \langle Vu, \pi(A)Vw \rangle_K + \langle u, GAw \rangle + \langle Ga' u, w \rangle
\]

for all \( u, w \in U \), and \( A \in \mathcal{A} \). Moreover, the linear map \( G : U \rightarrow \mathcal{H} \) can be chosen to be

\[
G(u) = L(|u \rangle \langle e |)e - \frac{1}{2} \langle e, L(|e \rangle \langle e |)e \rangle u, \quad u \in U.
\]

**Proof.** Follows immediately from Theorems 4.5 and 4.7.

We do not know if the map \( G \) that appears in Theorem 4.5 and Corollary 4.8 is closed. In Proposition 4.9, we define a linear operator \( \widehat{G} : U \rightarrow \mathcal{B}(\mathcal{H}) \) such that \( \widehat{G}(x)(e) = G(x) \) where \( e \) is the unit vector used in the definition of the linear space \( U \) and we study its closability.
Proposition 4.9. Let $L$ be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Suppose there exists a unit vector $e \in \mathcal{H}$ such that $|e\rangle\langle e| \in D(L)$. Let $U = \{x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A}\}$ and define $\hat{G} : U \to \mathcal{B}(\mathcal{H})$ by

$$\hat{G}(x)(v) = L(|x\rangle\langle e|)v - \frac{1}{2} (e, L(|e\rangle\langle e|)v)x.$$

Then, $\hat{G}$ is $(\|\cdot\|, \sigma$-weakly)-closable. Further, if we define $U_0 = \{x \in \mathcal{H} : |x\rangle\langle e| \in D(L)\}$, then $U \subseteq U_0$ and $\hat{G}$ defined on $U_0$ is $(\|\cdot\|, \sigma$-weakly)-closed.

Proof. Let $(x_n)_{n \geq 1} \subseteq U$ such that $x_n \to 0$ in norm and $\hat{G}(x_n) \to A \in \mathcal{B}(\mathcal{H})$ $\sigma$-weakly. Then, $|x_n\rangle\langle e| \in \mathcal{A} \subseteq D(L)$. We claim that $|x_n\rangle\langle e| \to 0$ $\sigma$-weakly. Indeed, let $(u_k)_{k \geq 1}, (v_k)_{k \geq 1} \subseteq \mathcal{H}$ such that $\sum_k \|u_k\|^2 < \infty$ and $\sum_k \|v_k\|^2 < \infty$. Then,

$$\sum_{k=1}^{\infty} \langle u_k, |x_n\rangle\langle e|v_k \rangle \leq \sum_{k=1}^{\infty} |\langle e, v_k \rangle\langle u_k, x_n \rangle| \leq \|x_n\| \left( \sum_{k=1}^{\infty} \|e\|^2 \|v_k\|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} \|u_k\|^2 \right)^{1/2} = c_1 \|x_n\|$$

and since $\|x_n\| \to 0$, we have that $|x_n\rangle\langle e| \to 0$ $\sigma$-weakly. Similarly, we claim that the sequence of bounded linear operators $v \mapsto \langle e, L(|e\rangle\langle e|)v \rangle x_n$ (simply denoted as $\langle e, L(|e\rangle\langle e|)\cdot \rangle x_n$) converges to 0 $\sigma$-weakly as $n \to \infty$. Indeed,

$$\sum_{k=1}^{\infty} \langle u_k, \langle e, L(|e\rangle\langle e|)v_k \rangle x_n \rangle \leq c_2 \|x_n\| \left( \sum_{k=1}^{\infty} \|u_k\|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} \|v_k\|^2 \right)^{1/2}.$$

Since $\langle e, L(|e\rangle\langle e|)\cdot \rangle x_n \to 0$ $\sigma$-weakly as $n \to \infty$ and $\hat{G}(x_n) \to A$ $\sigma$-weakly, we have that $L(|x_n\rangle\langle e|) \to A$ $\sigma$-weakly. Thus, since $L$ is $\sigma$-weakly closed on its domain $D(L)$ (see the work of Bratteli and Robinson,\textsuperscript{4} Theorem 3.1.10), and $|x_n\rangle\langle e| \to 0$ $\sigma$-weakly, we have that $A = L(0) = 0$ and therefore, $\hat{G}$ is closable. For the last statement of Proposition 4.9, suppose that $(x_n)_{n \geq 1} \subseteq U_0$, with $x_n \to x$ in norm and $\hat{G}(x_n) \to A$ $\sigma$-weakly. Repeating the above argument with $x_n - x$ in place of $x_n$, we obtain that $|x_n - x\rangle\langle e| \to 0$ $\sigma$-weakly (hence, $|x_n\rangle\langle e| \to |x\rangle\langle e|$ $\sigma$-weakly) and that $\langle e, L(|e\rangle\langle e|)\cdot \rangle (x_n - x) \to 0$ $\sigma$-weakly as $n \to \infty$; hence, $\langle e, L(|e\rangle\langle e|)\cdot \rangle x_n \to \langle e, L(|e\rangle\langle e|)\cdot \rangle x$ $\sigma$-weakly as $n \to \infty$. Since $\hat{G}(x_n) \to A$ $\sigma$-weakly, we obtain that

$$L(|x_n\rangle\langle e|) \to A + \frac{1}{2} \langle e, L(|e\rangle\langle e|)\cdot \rangle x$$

$\sigma$-weakly. Thus, since $L$ is $\sigma$-weakly closed on its domain $D(L)$, we obtain that $|x\rangle\langle e| = A + \frac{1}{2} \langle e, L(|e\rangle\langle e|)\cdot \rangle x$, i.e., $\hat{G}(x) = A$, which proves that $\hat{G}$ defined on $U_0$ is $(\|\cdot\|, \sigma$-weakly)-closed. \qed

As mentioned earlier, we will illustrate the form of the generator $L$ and discuss the subspace $U$ in several examples in Section VI, but first we would like to attempt obtaining an analogous result to that of Kraus’ work (Theorem 3.4).

V. AN ATTEMPT TO EXTEND KRAUS’ RESULT

Theorems 4.5 and 4.7 describe the form of the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Unfortunately, we do not have a result similar to Theorem 3.4 for the form of the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ which appears in the conclusion of Theorem 4.7. For the uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$, $\pi$ turns out to be a normal representation on $\mathcal{B}(\mathcal{H})$ and the map $V$ which appears in Theorem 3.3 turns out to be bounded.

This section is dedicated to proving, under suitable assumptions, continuity properties of the operators $V$ and $\phi$ which appear in Theorem 4.7 in the hopes of obtaining a dilation for $\phi$, similar to Theorem 3.4. While we do not achieve this, we get rather close and identify what we see is ultimately needed to finish. We also have some continuity results which are of interest in their own right.
Proposition 5.1. Let \( L \) be the generator of a QMS on the von Neumann algebra \( \mathcal{B}(\mathcal{H}) \). Further, suppose there exists a unit vector \( e \in \mathcal{H} \) such that \( |e\rangle\langle e| \in D(L) \). Let \( U = \{ x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A} \} \) and define \( G : U \to \mathcal{H} \) by

\[
Gx = L(|x\rangle\langle e|)e - \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle x
\]

and \( V : U \to \mathcal{K} \) by

\[
Vx = 1 \otimes x + N,
\]

where \( \mathcal{K} \) is the Hilbert space given in Theorem 4.7. Also, suppose that there exists \( C > 0 \) such that \( ||L(|x\rangle\langle e|)e|| \leq C||x|| \) for all \( x \in U \). (9)

Then, \( G \) is bounded on \( U \). If the map \( \phi \) of Theorem 4.7 satisfies the conclusion of Theorem 4.5, then the map \( V \) is bounded on \( U \) as well.

Proof. For \( x \in U \),

\[
||Gx|| = ||L(|x\rangle\langle e|)e - \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle x|| \leq C||x|| + \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle ||x|| \leq C'||x||
\]

and so, \( G \) is bounded on \( U \). Further, let \( x \in U \) Then,

\[
||Vx||^2 = ||1 \otimes x + N||^2 = (1 \otimes x, 1 \otimes x) = \phi(1)(x, x) = |\phi(1)(x, x)|.
\]

Hence, by the conclusion of Theorem 4.5, since \( L(1) = 0 \), we get that

\[
|\phi(1)(x, x)| = ||x, Gx - \langle Gx, x\rangle| \leq C||x||^2 + C||x||^2.
\]

Therefore, \( V \) is also bounded on \( U \). \( \square \)

The operator \( G \) of Proposition 5.1 is the same as in Theorem 4.5 and Corollary 4.8. The operator \( V \) of Proposition 5.1 is the same as in Theorem 4.7 and Corollary 4.8. Corollary 4.8 and Proposition 5.1 are used in the proof of the next result.

Proposition 5.2. Let \( L \) be the generator of a QMS on the von Neumann algebra \( \mathcal{B}(\mathcal{H}) \) and let \( \mathcal{A} \) denote its domain algebra. Further, suppose there exists a unit vector \( e \in \mathcal{H} \) such that \( |e\rangle\langle e| \in D(L) \). Let \( U = \{ x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A} \} \). Assume that (9) is valid and that \( \overline{U} = \mathcal{H} \). Then, there exist a linear map \( G : U \to \mathcal{H} \), a Hilbert space \( \mathcal{K} \), a linear map \( V : U \to \mathcal{K} \), and a unital \( \ast \)-representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) such that

\[
L(A) = V^\ast \pi(A)V + GA + AG^\ast
\]

for all \( A \in \mathcal{A} \). Further, define \( \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) by \( \psi(A) = GA + AG^\ast \). Then, \( \psi \) is \( \sigma \)-weakly \( \ast \)-continuous. Finally, the map \( \varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) defined by \( \varphi(A) = V^\ast \pi(A)V \) is \( \sigma \)-weakly \( \ast \)-weakly closable.

Remark 5.3. Note that the assumptions of Proposition 5.2 are rather strong since (10) implies that \( L \) is bounded on \( \mathcal{A} \) (but not necessarily on \( \mathcal{B}(\mathcal{H}) \)).

Proof of Proposition 5.2. By Corollary 4.8, there exist a linear map \( G : U \to \mathcal{H} \), a Hilbert space \( \mathcal{K} \), a linear map \( V : U \to \mathcal{K} \), and a unital \( \ast \)-representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) such that

\[
\langle x, L(A)y \rangle = \langle Vx, \pi(A)Vy \rangle + \langle x, Gay \rangle + \langle GA^\ast x, y \rangle
\]

for all \( A \in \mathcal{A} \) and \( x, y \in U \). By Proposition 5.1 and the assumption that \( \overline{U} = \mathcal{H} \), we see that

\[
\langle x, L(A)y \rangle = \langle x, V^\ast \pi(A)Vy \rangle + \langle x, Gay \rangle + \langle x, AG^\ast y \rangle
\]

for all \( A \in \mathcal{A} \) and \( x, y \in \mathcal{H} \). Thus, \( L(A) = V^\ast \pi(A)V + GA + AG^\ast \) for all \( A \in \mathcal{A} \). Let \( (A_\lambda)_\lambda \subseteq \mathcal{B}(\mathcal{H}) \) be a net such that \( A_\lambda \rightharpoonup A \) \( \sigma \)-weakly for some \( A \in \mathcal{B}(\mathcal{H}) \). Let \( (x_n)_{n\geq1}, (y_n)_{n\geq1} \subseteq \mathcal{H} \) such that
\[ \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \text{ and } \sum_{n=1}^{\infty} \|y_n\|^2 < \infty. \]

Then,
\[ \sum_{n=1}^{\infty} \langle x_n, \psi(A_\lambda)y_n \rangle = \sum_{n=1}^{\infty} (\langle x_n, GA_\lambda y_n \rangle + \langle x_n, A_\lambda G^* y_n \rangle) = \sum_{n=1}^{\infty} (\langle G^* x_n, A_\lambda y_n \rangle + \langle x_n, A_\lambda G^* y_n \rangle) \rightarrow \sum_{\lambda} (\langle G^* x_n, A y_n \rangle + \langle x_n, AG^* y_n \rangle) \]

since \( \sum_{n=1}^{\infty} \|G^* x_n\|^2 < \infty \) and \( \sum_{n=1}^{\infty} \|G^* y_n\|^2 < \infty \). So, we have that \( \psi \) is \( \sigma \)-weakly - \( \sigma \)-weakly continuous.

Next, let \( (A_\lambda)_{\lambda} \subseteq \mathcal{A} \) be a net such that \( A_\lambda \rightarrow 0 \) \( \sigma \)-weakly and \( \varphi(A_\lambda) \rightarrow B \) \( \sigma \)-weakly, for some \( B \in \mathcal{B}(\mathcal{H}) \), where \( \varphi(A) = V^* \pi(A)V \). Let \( (x_n)_{n \geq 1}, (y_n)_{n \geq 1} \subseteq \mathcal{H} \) such that \( \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \) and \( \sum_{n=1}^{\infty} \|y_n\|^2 < \infty \). Then,
\[ \sum_{n=1}^{\infty} \langle x_n, L(A_\lambda)y_n \rangle = \sum_{n=1}^{\infty} (\langle x_n, V^* \pi(A_\lambda)V y_n \rangle + \langle x_n, \psi(A_\lambda)y_n \rangle) = \sum_{n=1}^{\infty} \langle x_n, B y_n \rangle \]

since \( \psi \) is \( \sigma \)-weakly - \( \sigma \)-weakly continuous and \( \varphi(A_\lambda) \rightarrow B \) \( \sigma \)-weakly. Then, since \( L \) is \( \sigma \)-weakly-\( \sigma \)-weakly closed on its domain \( D(L) \) and therefore \( \sigma \)-weakly-\( \sigma \)-weakly closable on \( \mathcal{A} \), we have that \( B = L(0) = 0 \). So, we have that \( \varphi \) is \( \sigma \)-weakly - \( \sigma \)-weakly closable.

\[ \Box \]

If one assumes (9) but does not assume that \( \overline{U[\mathcal{H}]} = \mathcal{H} \), then the proof of Proposition 5.2 gives the following.

**Remark 5.4.** Consider the situation described in Proposition 5.2 without assuming that \( \overline{U[\mathcal{H}]} = \mathcal{H} \). Let
\[ F = \left\{ \sum_{n=1}^{\infty} \langle x_n \rangle \langle y_n \rangle : (x_n)_{n \geq 1}, (y_n)_{n \geq 1} \subseteq \overline{U[\mathcal{H}]} \text{ such that } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \, , \, \sum_{n=1}^{\infty} \|y_n\|^2 < \infty \right\}, \]

then \( F \subseteq L_1(\mathcal{H}) \) (the space of trace class operators on \( \mathcal{H} \)). Further, if we define \( \psi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}) \) by \( \psi(A) = GA + AG^* \), then under assumption (9), \( \psi \) is \( \sigma(\mathcal{A}, F) - \sigma(\mathcal{B}(\mathcal{H}), F) \) continuous.

We did not find an application of the above remark.

**Definition 5.5.** A pair \( (\pi, V) \) satisfying \( T(A) = V^* \pi(A)V \), where \( \pi \) is a representation on \( \mathcal{A} \) and \( V : U \rightarrow \mathcal{K} \) is called a **minimal Stinespring representation** if the set
\[ \{ \pi(A)Vu : A \in \mathcal{A}, \, u \in U \} \]

is total in \( \mathcal{K} \).

If we look back at Theorem 4.7 to the definitions of \( \phi, V, \) and \( \mathcal{K} \), it is easy to see that our \( (\pi, V) \) is a minimal Stinespring representation (as in the proof of the original result of Stinespring,\(^{25}\) Theorem 1). This will be used in the following result.

**Proposition 5.6.** Let \( L \) be the generator of a QMS on the von Neumann algebra \( \mathcal{B}(\mathcal{H}) \) and let \( \mathcal{A} \) denote its domain algebra. Suppose there exists a nonzero vector \( e \in \mathcal{H} \) such that \( |e\rangle \langle e| \in D(L) \). Let \( U = \{ x \in \mathcal{H} : |x\rangle \langle e| \in \mathcal{A} \} \). Also, suppose that (9) is valid and that \( \overline{U[\mathcal{H}]} = \mathcal{H} \). Then, the unital \( * \)-representation \( \pi : \mathcal{A} \rightarrow \mathcal{K} \) which appears in the statement of Proposition 5.2 is \( \sigma \)-weakly -\( \sigma \)-weakly closable.

**Proof.** Let \( (A_\lambda)_{\lambda} \subseteq \mathcal{A} \) be a net such that \( A_\lambda \rightarrow 0 \) \( \sigma \)-weakly and \( \pi(A_\lambda) \rightarrow B \) \( \sigma \)-weakly for some \( B \in \mathcal{B}(\mathcal{H}) \). Let \( C, D \in \mathcal{A} \). Then, it is trivial to see that \( C^*A_\lambda D \rightarrow 0 \) \( \sigma \)-weakly and \( \pi(C^*)\pi(A_\lambda)\pi(D) \rightarrow \pi(C^*)B\pi(D) \) \( \sigma \)-weakly. Since, by Proposition 5.1, \( V \) is bounded on \( \mathcal{H} \), we have
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\[ \varphi(C^*A_1D) = V^*\pi(C^*A_1D)V = V^*\pi(C^*)\pi(A_1)\pi(D)V \rightarrow V^*\pi(C)^*B\pi(D)V \]

\( \sigma \)-weakly. Well, \( \varphi \) is \( \sigma \)-weakly - \( \sigma \)-weakly closable by Proposition 5.2 and so, \( V^*\pi(C)^*B\pi(D)V = 0 \). Then, for any \( x, y \in \mathcal{H} \),

\[ \langle \pi(C)Vx, B\pi(D)Vy \rangle = \langle x, V^*\pi(C)^*B\pi(D)Vy \rangle = 0, \]

and, since \( \langle \pi, V \rangle \) is a minimal representation, \( B = 0 \). Therefore, \( \pi \) is closable. \( \square \)

In the application of the theorem of Kraus to the generators of uniformly continuous QMSs on \( \mathcal{B}(\mathcal{H}) \), \( \pi \) is a \( \sigma \)-weakly continuous unital \( * \)-representation so, for a cyclic vector \( \omega \in \mathcal{K} \), the map \( \mathcal{B}(\mathcal{H}) \ni A \mapsto \langle \omega, \pi(A)\omega \rangle \) is positive and \( \sigma \)-weakly continuous. Since we have a characterization of such maps, namely, positive trace-class operators acting on \( \mathcal{B}(\mathcal{H}) \) via the trace duality, we can conclude this map has the form

\[ \langle \omega, \pi(A)\omega \rangle = \sum_{n=1}^{\infty} \langle x_n, Ax_n \rangle, \]

where \( \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \). Unfortunately, if we replace \( \mathcal{B}(\mathcal{H}) \) with a (not necessarily closed) \( * \)-subalgebra \( \mathcal{A} \) and we only assume that the unital \( * \)-representation \( \pi : \mathcal{A} \rightarrow \mathcal{K} \) is (\( \sigma \)-weakly, \( \sigma \)-weakly)-closable (which is guaranteed by Proposition 5.6), we do not know the form of the map \( \mathcal{A} \ni A \mapsto \langle \omega, \pi(A)\omega \rangle \). This seems to be the missing ingredient in order to obtain an analogue result of Kraus for general QMS on \( \mathcal{B}(\mathcal{H}) \).

VI. EXAMPLES

We will now proceed to look at two examples of QMSs where we verify that their generators satisfy the form given by Corollary 4.8. We identify the linear maps \( G, V \), the representation \( \pi \), the Hilbert space \( \mathcal{K} \), and the linear subspace \( U \) of \( \mathcal{H} \) as in Corollary 4.8. Moreover, we prove that the subspace \( U \) is dense in \( \mathcal{H} \) in the first two examples.

Example 6.1 (The work of Parthasarathy\textsuperscript{21}, p. 258). Let \( (B_t)_{t \geq 0} \) be a standard Brownian motion defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and define \( T_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) by

\[ T_tA = \mathbb{E}\left[e^{iBtV} Ae^{-iBtV}\right], \]

where \( V \) is a self-adjoint operator on \( \mathcal{H} \). Then, \( (T_t)_{t \geq 0} \) is a QMS.

See the work of Parthasarathy\textsuperscript{21} (or the work of Ziemke\textsuperscript{28}) for the details as to why \( (T_t)_{t \geq 0} \) is a QMS and why the generator \( L \) of \( (T_t)_{t \geq 0} \) is given by

\[ L(A) = -\frac{1}{2} (adV)^2 A = -\frac{1}{2} (V^2 A + AV^2 - 2VAV). \]

Now, if \( V \) is unbounded, then the generator is given “formally” by the above equation, that is, \( L \) can be realized as a sesquilinear form where

\[ \langle u, L(A)v \rangle = \langle Vu, AVv \rangle + \langle u, -\frac{1}{2} V^2 Av \rangle + \langle -\frac{1}{2} V^2 A' u, v \rangle. \]

Also, the generator has the form given in Corollary 4.8 with \( G = -\frac{1}{2} V^2 \). If \( \mathcal{H} = L_2(\mathbb{R}) \) and \( V = i \frac{d}{dx} \), then let \( e(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \) and it is an easy exercise to see that

\[ U = \{ u \in L_2(\mathbb{R}) : |u\rangle\langle e| \in \mathcal{A} \} \supseteq \{ f \in L_2(\mathbb{R}) : f', f'' \in L_2(\mathbb{R}) \}, \]

and therefore, \( U \) is dense in \( L_2(\mathbb{R}) \).

Example 6.2 (The work of Arveson\textsuperscript{2} and similar examples produced in the works of Fagnola\textsuperscript{13} and Powers\textsuperscript{24}). Let \( \mathcal{H} = L_2[0, \infty) \) and define \( U_t : \mathcal{H} \rightarrow \mathcal{H} \) by

\[ (U_t g)(x) = \begin{cases} g(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise} \end{cases}. \]
Then, \((U_t)_{t \geq 0}\) is a strongly continuous semigroup of isometries whose generator \(\Delta\) is differentiation. Let \(f \in L_2(0, \infty)\) be what we get by normalizing \(u(x) = e^{-x} (i.e., f = \frac{u}{\|u\|})\) then define \(\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}\) by \(\omega(A) = \langle f, A f \rangle\). Define the completely positive maps \(\phi_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\), where

\[
\phi_t(A) = \omega(A)E_t + U_tAU_t^* 
\]

for all \(t \geq 0\), where \(E_t\) is the projection onto the subspace \(L_2(0,t) \subseteq L_2(0,\infty)\). Then, \((\phi_t)_{t \geq 0}\) is a QMS.

First note that for \(A \in \mathcal{B}(\mathcal{H})\),

\[
\omega(U_tAU_t^*) = \langle U_t^* f, AU_t^* f \rangle = \left(\frac{e^{-(t+i)}}{\|t\|}, A \left(\frac{e^{-(t+i)}}{\|t\|}\right)\right) = e^{-2t} \langle f, Af \rangle = e^{-2t} \omega(A),
\]

where the dots denote the variable of the function. We claim that \((\phi_t)_{t \geq 0}\) is a semigroup. First, we want to show that \(\omega(\phi_t(A)) = \omega(A)\) for all \(A \in \mathcal{B}(\mathcal{H})\). Indeed,

\[
\omega(\phi_t(A)) = \omega(\omega(A)E_t + U_tAU_t^*)
= \omega(A)\langle f, E_t f \rangle + \langle f, U_tAU_t^* f \rangle
= \omega(A)\langle f, (1-U_tU_t^*) f \rangle + e^{-2t} \omega(A) \quad \text{since} \quad E_t = 1-U_tU_t^*
= \omega(A)(1-e^{-2t}) + e^{-2t} \omega(A) = \omega(A).
\]

So, we have that \(\omega(\phi_t(A)) = \omega(A)\) for all \(A \in \mathcal{B}(\mathcal{H})\). Next, we want to show \(\phi_t\phi_s = \phi_{t+s}\). Let \(A \in \mathcal{B}(\mathcal{H})\). Then,

\[
\phi_s\phi_t(A) = \omega(\phi_t(A))E_s + U_s(\omega(A)E_t + U_tAU_t^*)U_s^* = \omega(A)(E_s + U_sE_tU_s^*) + U_{s+t}A(U_{s+t})^*
\]

and, since \(E_{s+t} = E_s + U_sE_tU_s^*\), we have that

\[
\phi_s\phi_t(A) = \omega(A)E_{s+t} + U_{s+t}A(U_{s+t})^* = \phi_{s+t}(A).
\]

So, we have that \((\phi_t)_{t \geq 0}\) is a QMS. If \(L\) denotes the generator of \((\phi_t)_{t \geq 0}\) and \(D(L)\) denotes the domain of \(L\), then

\[
L(A) = \sigma - \text{weak} - \lim_{t \to 0} \frac{1}{t} (\phi_t(A) - A)
= \sigma - \text{weak} - \lim_{t \to 0} \frac{\omega(A)E_t + U_tAU_t^* - A}{t}
\quad \text{for all } A \in D(L). \tag{11}
\]

By Example 3.1, the generator of the QMS \((A \mapsto U_tAU_t^*)_{t \geq 0}\) is equal to \(\sigma_\Delta\), where \(\sigma_\Delta(A) = \Delta A + A \Delta^*\). For \(A \in D(L) \cap D(\sigma_\Delta)\), if \(u,v \in L_2[0,\infty)\) such that \(u',v' \in L_2[0,\infty)\) and \(u\) and \(v\) are continuous at zero, then

\[
\langle u, L(A)v \rangle = \omega(A)\overline{u(0)}v(0) + \langle u, \Delta Au \rangle + \langle \Delta A^*u, v \rangle
\]

and one can check that the map \((A,u,v) \mapsto \omega(A)\overline{u(0)}v(0)\) is completely positive for appropriately chosen \(A \in D(L)\) and \(u,v \in L_2[0,\infty)\). By (11), if a bounded operator \(A\) belongs to \(D(\sigma_\Delta)\) and the kernel of \(\omega\), (i.e., \(\omega(A) = 0\)), then \(A \in D(L)\). Now fix a normalized vector \(e \in L_2[0,\infty)\) such that \(D(e) \subseteq L_2[0,\infty)\) and \((e, f) = 0\) (for example, take \(e\) to be what we get by normalizing \(e_0 = g - \langle f, g \rangle f\), where \(g\) is any function in \(L_2[0,\infty)\) such that \(g' \in L_2[0,\infty)\) and \(\|g\| = 1\) and use this vector \(e\) to define the subspace \(U\) of Corollary 4.8. Using the fact that \(\ker \omega \cap D(\sigma_\Delta) \subseteq D(L)\) is easy to verify that for all \(x \in L_2[0,\infty)\) \(\Delta x \in L_2[0,\infty)\) and \((x, f) = 0\), we have that the following three conditions are satisfied: \(|x(e)| \in D(L), |x(e)|(|x(e)|)^* = \|x\|^2|x(e)| \in D(L), \) and \((|x(e)|)^*|x(e)| = \|x\|^2|e|e|e| \in D(L).\) Thus, by Definition 4.4, \(|x(x)| \in \mathcal{A}\), where \(\mathcal{A}\) denotes the domain algebra of \(L\). Hence,

\[
\{x \in L_2[0,\infty) : \Delta x \in L_2[0,\infty)\} \quad \text{and} \quad \langle x, f \rangle = 0 \subseteq U. \tag{12}
\]

Arveson\(^2\) the proposition of p. 75, proves that the strong operator closure \(\overline{\mathcal{A}}_{\text{SOT}}\) of the domain algebra is equal to the set of bounded operators \(A\) such that both \(A\) and its adjoint \(A^*\) have \(f\) as an
eigenvector (necessarily corresponding to complex conjugate eigenvalues). Thus, for \( x \in L^2[0, \infty) \), if \( A = |x\rangle\langle e| \in \mathcal{A} \), then \( \langle f, e \rangle = 0 \). Therefore,

\[
U \subseteq \{ x \in L^2[0, \infty) : \langle x, f \rangle = 0 \}.
\]

(13)

We do not have a more precise description of \( U \) besides (12) and (13). Equation (13) shows that \( U \) is not dense in \( \mathcal{H} \). Note that the domain algebra \( \mathcal{A} \) contains operators which are not in the kernel of \( \omega \) (since \( \|f\|_2 \in \mathcal{A} \) by Arveson,\(^2\) the proposition of p. 75). Hence, the operator \( V \) and the unital \( * \)-representation \( \pi \) which appear in the statement of Corollary 4.8 are nonzero. The operator \( G \) which appears in the statement of Corollary 4.8 is not necessarily equal to the generator \( \mathcal{D} \) of \( (U_t)_{t \geq 0} \). Formulas for \( V, \pi, \) and \( G \) are given in Corollary 4.8 and Theorems 4.5 and 4.7 and we do not know simpler formulas for this particular example.