Quantum Info Seminar (Steve Fenner)

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What is a quantum channel? (Sept. 29, 2023)

Represents some (any) quantum process that physically realizable (in principle).

Examples:

- A quantum computation
- Unitary evolution of an isolated system
- Measurements
- Q. & classical communication
- model errors or noise
- combinations of the above

Notation: A $\mathbb{C}$-space $\mathcal{H}$ is a finite dimensional Hilbert with an inner product $\langle \cdot , \cdot \rangle$ — positive definite $\langle u,u \rangle > 0$ if $u \neq 0$

$\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ — linear in 2nd arg — conjugate symmetric

and an orthonormal basis $\{e_1, \ldots, e_n\}$ $(n = \dim(\mathcal{H}))$

$$\langle e_i, e_j \rangle = \delta_{ij}$$

For $\mathbb{C}$-spaces $\mathcal{H}, \mathcal{J}$: $L(\mathcal{H}, \mathcal{J}) = \{A : \mathcal{H} \to \mathcal{J} \text{ is linear}\}$

$L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$

Adjoints: $A \in L(\mathcal{H}, \mathcal{J})$. $A^* \in L(\mathcal{J}, \mathcal{H})$ is unique map such that, $\forall u \in \mathcal{H}\ \forall v \in \mathcal{J}$, $\langle v, Au \rangle_{\mathcal{J}} = \langle A^* v, u \rangle_{\mathcal{H}}$

<u>Def:</u> A quantum channel (from $\mathcal{H}$ to $\mathcal{J}$) is a linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ $\quad [\Phi \in L(L(\mathcal{H}), L(\mathcal{J}))]$
that is

- trace-preserving
- completely positive

"trace preserving" means $\operatorname{tr}(\Phi(A)) = \operatorname{tr} A \quad \forall A \in L(\mathcal{H})$

<u>Def:</u> Φ is <u>positive</u> if $\forall A \in L(\mathcal{H})$, if $\underbrace{A \geq 0}_{\text{A is positive (semidef)}}$ then $\Phi(A) \geq 0$

Let K be any $\mathbb{C}$-space. Can form the linear map

$$\Phi \otimes \mathbb{1}_{L(K)} : L(\mathcal{H} \otimes K) \to L(\mathcal{J} \otimes K)$$

defined so that $\forall A \in L(\mathcal{H})$ and $B \in L(K)$,

$$(\Phi \otimes \mathbb{1}_{L(K)})(A \otimes B) = \Phi(A) \otimes B$$

[If Φ is trace-preserving, then so is $\Phi \otimes \mathbb{1}_{L(K)}$ routine proof]

<u>Def:</u> $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ is <u>completely positive</u> if $\Phi \otimes \mathbb{1}_{L(K)}$ is positive for every $\mathbb{C}$-space K.

<u>Example</u> of a positive operator that is not completely positive: transpose operator $\quad [A \in \mathbb{C}^{2\times 2}]$

$$\Phi(A) = A^T$$

$\Phi \otimes \mathbb{1}_{\mathbb{C}^{2\times 2}}$ is <u>not</u> positive

$L(\mathcal{H},\mathcal{J})$ is a $\mathbb{C}$-space! $\forall A,B \in L(\mathcal{H},\mathcal{J})$

$$\langle A, B\rangle := tr(A^* B)$$

Examples:

Unitary Channel: $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Phi(A) = UAU^* \qquad [U \in L(\mathcal{H}) \text{ is unitary}]$$

describes any process on an isolated physical system. E.g. $\mathbb{1}_{L(\mathcal{H})}$ is a unitary channel ($U = \mathbb{1}_{\mathcal{H}}$) that causes no change (represents an ideal (noiseless) comm. channel, or perfectly preserved memory).

Replacement Channel: Fix $\sigma \in L(\mathcal{J})$ density operator
$\sigma \geq 0$ & $tr\,\sigma = 1$

Define, for all $A \in L(\mathcal{H})$

$$\Phi(A) = (Tr\,A)\,\sigma \qquad [\text{so } \Phi(\rho) = \sigma \ \ \text{any density operator } \rho]$$

State Preparation: $\Phi : \underbrace{L(\mathbb{C})}_{=\mathbb{C}} \to L(\mathcal{H})$ defined by

$\forall \alpha \in \mathbb{C}$, $\Phi(\alpha) = \alpha\rho$ [some fixed density operator ρ]

POVM (Positive operator-valued Measure):

Most general measurement on a quantum system where post-measurement state is ignored.

Set $\{M_1, \ldots, M_k\}$, where each $M_i \in L(\mathcal{H})$, $M_i \geq 0$ and $\sum_{i=1}^{k} M_i = \mathbb{1}_{\mathcal{H}}$.

The corresponding POVM is the channel $\Phi : L(\mathcal{H}) \to L(\mathbb{C}^k)$

$$\Phi(A) = \sum_{i=1}^{k} \mathrm{Tr}(AM_i) E_{ii} = \underbrace{\begin{bmatrix} \mathrm{Tr}(AM_1) & & 0 \\ & \ddots & \\ 0 & & \mathrm{Tr}(AM_k) \end{bmatrix}}_{\text{"classical" state}}$$

$$\Phi : L(\mathcal{H}_1) \to L(\mathcal{J}_1)$$
$$\Psi : L(\mathcal{H}_2) \to L(\mathcal{J}_2)$$
channels

implies

$$\Phi \otimes \Psi : L(\mathcal{H}_1 \otimes \mathcal{H}_2) \to L(\mathcal{J}_1 \otimes \mathcal{J}_2)$$

is a channel

Trace Map: $\mathrm{Tr} : L(\mathcal{H}) \to \underbrace{L(\mathbb{C})}_{=\mathbb{C}}$ is a channel

"destroy $\mathcal{H}$"
"discard $\mathcal{H}$"
"ignore $\mathcal{H}$"

Partial Trace: $\mathrm{Tr}_{\mathcal{K}} : \cancel{L(\mathcal{H})}\ L(\mathcal{H}\otimes\mathcal{K}) \to L(\mathcal{H})$ $\quad$ $\mathcal{K}$ is a $\mathbb{C}$-space

$$\boxed{\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr}} : L(\mathcal{H} \otimes \mathcal{K}) \to L(\mathcal{H}) \quad \text{(channel)}$$

"ignore system $\mathcal{K}$"
"trace out $\mathcal{K}$"

$$(\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr})(A \otimes B) = (\mathrm{tr}\, B) A \qquad \begin{array}{l} A \in L(\mathcal{H}) \\ B \in L(\mathcal{K}) \end{array}$$

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Steve Fenner

What is a Quantum Channel - Part 2.

Partial trace (correct this time):

$\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces $\quad \mathrm{Tr}_{\mathcal{J}} : L(\mathcal{H}\otimes\mathcal{J}) \to L(\mathcal{H})$

$$\mathrm{Tr}_{\mathcal{J}} := \mathbb{1}_{L(\mathcal{H})} \otimes \mathrm{tr}$$

$$\mathrm{Tr}_{\mathcal{J}}(A\otimes B) = (\mathrm{tr}\, B)A$$

Recall transpose op $L(\mathcal{H}) \to L(\mathcal{H})$
$X \mapsto X^T$

trace-preserving, positive but not completely positive. Counter-example:

$$\mathcal{H} = \mathcal{J} = \mathbb{C}^2$$

$X := uu^*$ where $u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$

$$X = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \geq 0$$

$$\left(\mathbb{1}_{L(\mathbb{C}^2)} \otimes \text{transpose}\right)(X) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Apply to $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ — conjugate on both sides get -2

<u>Adjoints</u>: For any linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ define

$\Phi^* : L(\mathcal{J}) \to L(\mathcal{H})$ s.t. $\forall A \in L(\mathcal{J})$ $\forall B \in L(\mathcal{H})$

$$\langle A, \Phi(B) \rangle = \langle \Phi^*(A), B \rangle$$

<u>Def</u>: Φ is <u>unital</u> if $\Phi(\mathbb{1}) = \mathbb{1}$

<u>Fact</u>: $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ linear.

(1) Φ is completely positive $\Longleftrightarrow$ Φ^* is completely pos

(2) Φ is trace preserving $\Longleftrightarrow$ Φ^* is unital.

<u>Proof of (2)</u>: $(\Rightarrow)$ $\forall B \in L(\mathcal{H})$

$$\langle \Phi^*(\mathbb{1}_{\mathcal{J}}), B \rangle = \langle \mathbb{1}_{\mathcal{J}}, \Phi(B) \rangle = \operatorname{tr} \Phi(B) = \operatorname{tr} B$$

$= \langle \mathbb{1}_{\mathcal{H}}, B \rangle$ $\qquad \therefore \Phi^*(\mathbb{1}_{\mathcal{J}}) = \mathbb{1}_{\mathcal{H}}$. //

$(\Longleftarrow)$: $\forall A \in L(\mathcal{H})$, $\operatorname{tr}\Phi(A) = \langle \mathbb{1}_{\mathcal{J}}, \Phi(A) \rangle$

$= \langle \Phi^*(\mathbb{1}_{\mathcal{J}}), A \rangle = \langle \mathbb{1}_{\mathcal{H}}, A \rangle = \operatorname{tr} A$. //

<u>Lemma</u>: $A \in L(\mathcal{H})$. $A \geq 0$ iff $\langle A, B \rangle \geq 0$ for all $B \geq 0$.

<u>Lemma</u>: Φ positive $\Longrightarrow$ Φ^* is positive.

<u>Proof</u>: $\forall A, B \geq 0$ $\langle \Phi^*(A), B \rangle = \langle A, \Phi(B) \rangle \geq 0$ //

<u>Cor</u>: ((1) of the Fact): Φ completely positive $\Longrightarrow$ Φ^* " " .

Pf: Φ completely positive means $\Phi \otimes \mathbb{1}_K$ pos. $\forall \mathbb{C}$-space K

Then $\Phi^* \otimes \mathbb{1}_K = (\underbrace{\overset{\text{comp pos}}{\Phi} \otimes \mathbb{1}_K}_{\text{pos}})^*$ $\quad\square$

Representations of Quantum Channels

Operator Sum Rep. (Kraus Rep):

Thm: A linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ is a channel iff $\exists K_1, \dots, K_n \in L(\mathcal{H}, \mathcal{J})$,

$$\sum_{i=1}^{n} K_i^* K_i = \mathbb{1}_{\mathcal{H}} \quad \longleftarrow \text{completeness condition}$$

and, $\forall A \in L(\mathcal{H})$,

$$\Phi(A) = \sum_{i=1}^{n} K_i A K_i^* .$$

The K_i are called Kraus operators.

Kraus rank of Φ = least n such that $K_1, \dots, K_n$ exist.

Φ Kraus rank $1 \iff \Phi$ is unitary channel

Choi Representation:

Given linear $\Phi : L(\underset{n\text{-dim}}{\mathcal{H}}) \to L(\underset{m\text{-dim}}{\mathcal{K}})$

define

$$J(\Phi) := \sum_{\substack{i,j \in [n] \\ [n] = \{1,\dots,n\}}} \Phi(E_{i,j}) \otimes E_{i,j}$$

[$\{E_{i,j} := e_i e_j^T\}$ standard basis for $L(\mathcal{H})$.]
(mn × mn matrix)

$J(\Phi) \in L(K \otimes \mathcal{H})$

<u>Fact</u>: (1) Φ completely positive $\iff J(\Phi) \geq 0$

digress

$$\sum_{i,j} E_{i,j} \otimes \Phi(E_{i,j}) = \left[\begin{array}{c|c|c} \Phi(E_{11}) & \Phi(E_{12}) & \cdots \\ \hline \Phi(E_{21}) & \Phi(E_{22}) & \\ \hline \vdots & \vdots & \ddots \end{array}\right]$$

(2) Φ is trace-preserving $\iff Tr_K(J(\Phi)) = \mathbb{1}_{\mathcal{H}}$

Proof of (2):

$$\mathrm{Tr}_K(J(\Phi)) = \mathrm{Tr}_K\Big(\sum_{i,j} \Phi(E_{i,j}) \otimes E_{i,j}\Big)$$

always true

$$= \sum_{i,j} \mathrm{Tr}_K\Big(\underbrace{\Phi(E_{i,j})}_{\in L(K)} \otimes \underbrace{E_{i,j}}_{\in L(\mathcal{H})}\Big)$$

$\sum_i E_{ii} = \mathbb{1}_{\mathcal{H}}$

$$= \sum_{i,j} \mathrm{tr}\big(\Phi(E_{i,j})\big) E_{ij} = \sum_{i,j} \delta_{ij} E_{i,j}$$

iff $\mathrm{tr}(\Phi(E_{i,j})) = \delta_{ij} \quad \forall i,j$

iff $\mathrm{tr}(\Phi(E_{i,j})) = \mathrm{tr}\, E_{ij}$

iff Φ is trace preserving

Summary $\mathrm{Tr}_K(J(\Phi)) = \mathbb{1}_{\mathcal{H}}$ iff Φ is trace-preserving.

$\boxtimes$

'Stinespring Rep:

Any linear $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$, there exists K and $A, B \in L(\mathcal{H}, \mathcal{J} \otimes K)$ such that, $\forall X \in L(\mathcal{H})$

$$\Phi(X) = \mathrm{Tr}_K\big(\overbrace{AXB^*}^{\in L(\mathcal{J}\otimes K)}\big)$$

<u>Fact:</u> Φ is a channel if $\exists K$, $\underset{\text{isometry}}{A \in L(\mathcal{H}, \mathcal{J}\otimes K)}$ such that, $\forall X \in L(\mathcal{H})$

$$\Phi(X) = \mathrm{Tr}_K(AXA^*)$$

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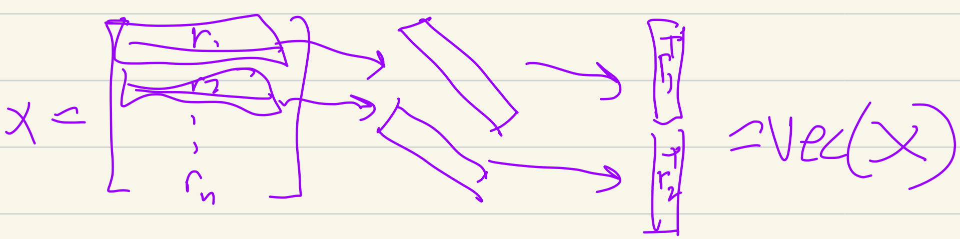
Connections between the reps

Prop: Let $\{A_i\}_{1\le i\le d}$ and $\{B_i\}_{1\le i\le d}$ be such that each $A_i, B_i \in L(\mathcal{H}, \mathcal{J})$ and let $\mathcal{K}$ be any d-dimensional $\mathbb{C}$-space. TFAE for any linear $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$:

1. $\Phi(X) = \sum_{i=1}^{d} A_i X B_i^* \qquad \forall X \in L(\mathcal{H})$
2. $J(\Phi) = \sum_{i=1}^{d} \mathrm{vec}(A_i)\,\mathrm{vec}(B_i)^*$

[where $\mathrm{vec} : L(\mathcal{H}, \mathcal{J}) \to \mathcal{J} \otimes \mathcal{H}$ linear such that $\mathrm{vec}(E_{ij}) = e_i \otimes e_j$



$\langle A, B\rangle = \langle \mathrm{vec}(A), \mathrm{vec}(B)\rangle$

3. $\Phi(X) = \mathrm{Tr}_{\mathcal{K}}(AXB^*)$ where

$$A = \sum_{i=1}^{d} A_i \otimes e_i$$

$$B = \sum_{i=1}^{d} B_i \otimes e_i$$

$e_i \in \mathcal{K}$
ith basis vector

More examples of channels

- Completely depolarizing channel $\Omega : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Omega(X) := (\mathrm{Tr}\, X)\, \mathbb{1}_{\mathcal{H}} / \dim \mathcal{H}$$

A partially depolarizing channel is of the form,

$$p\Omega + (1-p)\mathbb{1}_{L(\mathcal{H})} \qquad p \in [0,1]$$

$$J(\Omega) = \frac{\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{H}}}{\dim \mathcal{H}}$$

- Completely Dephasing Channel $\Delta : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Delta(X) := \sum_{i=1}^{\dim \mathcal{H}} [X]_{ii} E_{ii}$$

"set all off-diagonal elements to 0"

$$J(\Delta) = \sum_{i=1}^{\dim \mathcal{H}} E_{ii} \otimes E_{ii}$$

$$\Delta(X) = \sum_i E_{ii} X E_{ii}^*$$

$$= \mathrm{Tr}_{\mathcal{K}}\left(A X A^*\right)$$

where $A = \sum_{i=1}^{\dim \mathcal{H}} (e_i \otimes e_i) e_i^*$

& $\mathcal{K} = \mathcal{H}$.

Mixed Unitary Channel — convex combo of unitary channels.

In 2 dims,

$$\Omega(\rho) := \frac{1}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right)$$

Norms of superoperators $A \in L(\mathcal{H}, \mathcal{J})$

$$\|A\|_1 := \operatorname{Tr}\sqrt{A^* A}$$

induces a trace norm on linear operators $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$:

$$\|\Phi\|_1 := \max\{\|\Phi(X)\|_1 : \|X\|_1 \le 1\}$$

$$\left[= \max_{\|u\|=\|v\|=1} \|\Phi(uv^*)\|_1 \underset{\uparrow}{=} \max_{\|u\|=1} \operatorname{Tr}\left(\Phi(uu^*)\right) \right]$$

Thm of Russo-Dye

<u>Cor</u>: If Φ is pos, trace-preserving, then $\|\Phi\|_1 = 1$
& $\|\Phi(X)\|_1 \le \|X\|_1$

Completely Bounded Trace Norm ("diamond norm")

$$|||\Phi|||_1 := \|\Phi \otimes \mathbb{1}_{L(\mathcal{H})}\|_1$$

Key properties: $|||\Phi \otimes \mathbb{1}_{L(\mathcal{K})}|||_1 = |||\Phi|||_1$

$|||\Psi \circ \Phi|||_1 \le |||\Psi|||_1 \, |||\Phi|||_1$

$|||\Psi_1 \circ \Psi_0 - \Phi_1 \circ \Phi_0|||_1 \le |||\Psi_0 - \Phi_0|||_1 + |||\Psi_1 - \Phi_1|||_1$

$|||\Phi \otimes \Psi|||_1 = |||\Phi|||_1 \, |||\Psi|||_1$