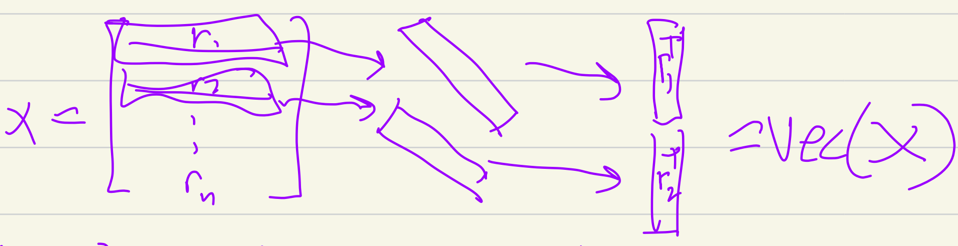
Quantum Seminar, Friday the 13th (Oct.)

Connections between the reps

<u>Prop</u>: Let $\{A_i\}_{1 \le i \le d}$ and $\{B_i\}_{1 \le i \le d}$ be such that each $A_i, B_i \in L(\mathcal{H}, \mathcal{J})$ and let $\mathcal{K}$ be any d-dimensional $\mathbb{C}$-space. TFAE for any linear $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$:

1. $\Phi(X) = \sum_{i=1}^{d} A_i X B_i^* \quad \forall X \in L(\mathcal{H})$
2. $J(\Phi) = \sum_{i=1}^{d} \text{vec}(A_i)\,\text{vec}(B_i)^*$

[where $\text{vec} : L(\mathcal{H}, \mathcal{J}) \to \mathcal{J} \otimes \mathcal{H}$ linear such that $\text{vec}(E_{ij}) = e_i \otimes e_j$



$\langle A, B \rangle = \langle \text{vec}(A), \text{vec}(B) \rangle$

3. $\Phi(X) = \mathrm{Tr}_{\mathcal{K}}(AXB^*)$ where

$$A = \sum_{i=1}^{d} A_i \otimes e_i$$

$$B = \sum_{i=1}^{d} B_i \otimes e_i$$

$e_i \in \mathcal{K}$
orth basis vector

More examples of channels

- Completely depolarizing channel $\Omega : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Omega(X) := (\mathrm{Tr}\, X)\, \mathbb{1}_{\mathcal{H}} / \dim \mathcal{H}$$

A partially depolarizing channel is of the form,

$$p\Omega + (1-p)\mathbb{1}_{L(\mathcal{H})} \qquad p \in [0,1]$$

$$J(\Omega) = \frac{\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{H}}}{\dim \mathcal{H}}$$

- Completely Dephasing Channel $\Delta : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Delta(X) := \sum_{i=1}^{\dim \mathcal{H}} [X]_{ii} E_{ii}$$

"set all off-diagonal elements to 0"

$$J(\Delta) = \sum_{i=1}^{\dim \mathcal{H}} E_{ii} \otimes E_{ii}$$

$$\Delta(X) = \sum_i E_{ii} X E_{ii}^*$$

$$= \mathrm{Tr}_{\mathcal{K}}\left(A X A^*\right)$$

where $A = \sum_{i=1}^{\dim \mathcal{H}} (e_i \otimes e_i) e_i^*$

& $\mathcal{K} = \mathcal{H}$.

Mixed Unitary Channel — convex combo of unitary channels.

In 2 dims,

$$\Omega(\rho) := \frac{1}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right)$$

Norms of superoperators $A \in L(\mathcal{H}, \mathcal{J})$

$$\|A\|_1 := \mathrm{Tr}\sqrt{A^* A}$$

induces a trace norm on linear operators $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$:

$$\|\Phi\|_1 := \max\{\|\Phi(X)\|_1 : \|X\|_1 \le 1\}$$

$$\left[= \max_{\|u\|=\|v\|=1} \|\Phi(uv^*)\|_1 \underset{\substack{\uparrow \\ \text{Thm of Russo-Dye}}}{=} \max_{\|u\|=1} \mathrm{Tr}\left(\Phi(uu^*)\right) \right]$$

Cor: If Φ is pos, trace-preserving, then $\|\Phi\|_1 = 1$

& $\|\Phi(X)\|_1 \le \|X\|_1$

Completely Bounded Trace Norm ("diamond norm")

$$|||\Phi|||_1 := \|\Phi \otimes \mathbb{1}_{L(\mathcal{H})}\|_1$$

Key properties: $|||\Phi \otimes \mathbb{1}_{L(\mathcal{K})}|||_1 = |||\Phi|||_1$

$|||\Psi \circ \Phi|||_1 \le |||\Psi|||_1 \, |||\Phi|||_1$

$|||\Psi_1 \circ \Psi_0 - \Phi_1 \circ \Phi_0|||_1 \le |||\Psi_0 - \Phi_0|||_1 + |||\Psi_1 - \Phi_1|||_1$

$|||\Phi \otimes \Psi|||_1 = |||\Phi|||_1 \, |||\Psi|||_1$