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What is a Quantum Channel - Part 2.

Partial trace (correct this time):

$\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces $\quad \mathrm{Tr}_{\mathcal{J}} : L(\mathcal{H} \otimes \mathcal{J}) \to L(\mathcal{H})$

$$\mathrm{Tr}_{\mathcal{J}} := \mathbb{1}_{L(\mathcal{H})} \otimes \mathrm{tr}$$

$$\mathrm{Tr}_{\mathcal{J}}(A \otimes B) = (\mathrm{tr}\, B) A$$

Recall transpose op $L(\mathcal{H}) \to L(\mathcal{H})$
$\qquad X \mapsto X^T$

trace-preserving, positive but not completely positive. Counter-example:

$$\mathcal{H} = \mathcal{J} = \mathbb{C}^2$$

$X := uu^*$ where $u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$

$$X = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \geq 0$$

2

$$\left(\mathbb{1}_{L(\mathbb{C}^2)} \otimes \text{transpose}\right)(X) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Apply to $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ — conjugate on both sides, get -2

<u>Adjoints</u>: For any linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ define

$\Phi^* : L(\mathcal{J}) \to L(\mathcal{H})$ s.t. $\forall A \in L(\mathcal{J})$, $\forall B \in L(\mathcal{H})$

$$\langle A, \Phi(B) \rangle = \langle \Phi^*(A), B \rangle$$

<u>Def</u>: Φ is <u>unital</u> if $\Phi(\mathbb{1}) = \mathbb{1}$

<u>Fact</u>: $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ linear.

(1) Φ is completely positive $\Longleftrightarrow$ Φ^* is completely PS

(2) Φ is trace preserving $\Longleftrightarrow$ Φ^* is unital.

<u>Proof of (2)</u>: $(\Rightarrow)$ $\forall B \in L(\mathcal{H})$

$$\langle \Phi^*(\mathbb{1}_{\mathcal{J}}), B \rangle = \langle \mathbb{1}_{\mathcal{J}}, \Phi(B) \rangle = \operatorname{tr} \Phi(B) = \operatorname{tr} B$$

$= \langle \mathbb{1}_{\mathcal{H}}, B \rangle$ $\qquad \therefore \Phi^*(\mathbb{1}_{\mathcal{J}}) = \mathbb{1}_{\mathcal{H}}$. //

$(\Longleftarrow)$: $\forall A \in L(\mathcal{H})$, $\operatorname{tr} \Phi(A) = \langle \mathbb{1}_{\mathcal{J}}, \Phi(A) \rangle$

$= \langle \Phi^*(\mathbb{1}_{\mathcal{J}}), A \rangle = \langle \mathbb{1}_{\mathcal{H}}, A \rangle = \operatorname{tr} A$. //

Lemma: $A \in L(\mathcal{H})$. $A \geq 0$ iff $\langle A, B \rangle \geq 0$ for all $B \geq 0$.

Lemma: Φ positive $\Longrightarrow$ Φ^* is positive.

Proof: $\forall A, B \geq 0$ $\langle \Phi^*(A), B \rangle = \langle A, \Phi(B) \rangle \geq 0$ //

Cor: ((1) of the Fact): Φ completely positive $\Longrightarrow$ Φ^* completely positive.

Pf: Φ completely positive means $\Phi \otimes \mathbb{1}_{\mathcal{K}}$ pos. $\forall \mathbb{C}$-space $\mathcal{K}$

Then $\Phi^* \otimes \mathbb{1}_{\mathcal{K}} = (\underbrace{\overset{\text{comp pos}}{\Phi} \otimes \mathbb{1}_{\mathcal{K}}}_{\text{pos}})^*$ ▨

Representations of Quantum Channels

Operator Sum Rep. (Kraus Rep):

Thm: A linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ is a channel iff $\exists K_1, \dots, K_n \in L(\mathcal{H}, \mathcal{J})$,

$$\sum_{i=1}^{n} K_i^* K_i = \mathbb{1}_{\mathcal{H}} \quad \longleftarrow \text{completeness condition}$$

and, $\forall A \in L(\mathcal{H})$,

$$\Phi(A) = \sum_{i=1}^{n} K_i A K_i^* .$$

The K_i are called Kraus operators.

Kraus rank of Φ = least n such that $K_1, \dots, K_n$ exist.

Φ Kraus rank $1 \iff \Phi$ is unitary channel

Choi Representation: Given linear $\Phi : L(\mathcal{H}) \to L(\mathcal{K})$ ($\mathcal{H}$: n-dim, $\mathcal{K}$: m-dim)

define

$$J(\Phi) := \sum_{\substack{i,j \in [n] \\ \parallel \\ \{1,\ldots,n\}}} \Phi(E_{i,j}) \otimes E_{i,j}$$

[$\{E_{i,j} := e_i e_j^T\}$ standard basis for $L(\mathcal{H})$.]

(mn × mn matrix)

$J(\Phi) \in L(K \otimes \mathcal{H})$

<u>Fact</u>: (1) Φ completely positive $\Longleftrightarrow J(\Phi) \geq 0$

digress

$$\sum_{i,j} E_{i,j} \otimes \Phi(E_{i,j}) = \left[\begin{array}{c|c|c} \Phi(E_{11}) & \Phi(E_{12}) & \cdots \\ \hline \Phi(E_{21}) & \Phi(E_{22}) & \\ \hline \vdots & \vdots & \ddots \end{array}\right]$$

(2) Φ is trace-preserving $\Longleftrightarrow \mathrm{Tr}_K(J(\Phi)) = \mathbb{1}_{\mathcal{H}}$

Proof of (2):

always true

$$\operatorname{Tr}_K\big(J(\Phi)\big) = \operatorname{Tr}_K\Big(\sum_{i,j} \Phi(E_{i,j}) \otimes E_{i,j}\Big)$$

$$= \sum_{i,j} \operatorname{Tr}_K\Big(\underbrace{\Phi(E_{i,j})}_{\in L(K)} \otimes \underbrace{E_{i,j}}_{\in L(\mathcal{H})}\Big)$$

$$= \sum_{i,j} \operatorname{tr}\big(\Phi(E_{i,j})\big) E_{ij} = \sum_{i,j} \delta_{ij} E_{i,j} = \sum_i E_{ii} = \mathbb{1}_{\mathcal{H}}$$

iff $\operatorname{tr}\big(\Phi(E_{i,j})\big) = \delta_{ij} \quad \forall i,j$

iff $\operatorname{tr}\big(\Phi(E_{i,j})\big) = \operatorname{tr} E_{ij}$

iff Φ is trace preserving

Summary $\operatorname{Tr}_K\big(J(\Phi)\big) = \mathbb{1}_{\mathcal{H}}$ iff Φ is trace-preserving. $\boxtimes$

Stinespring Rep:

Any linear $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$,
there exists $\mathcal{K}$ and $A, B \in L(\mathcal{H}, \mathcal{J} \otimes \mathcal{K})$
such that, $\forall X \in L(\mathcal{H})$

$$\Phi(X) = \mathrm{Tr}_{\mathcal{K}}\big(\overbrace{AXB^*}^{\in L(\mathcal{J} \otimes \mathcal{K})}\big)$$

<u>Fact:</u> Φ is a channel if $\exists \mathcal{K}$, $\underbrace{A \in L(\mathcal{H}, \mathcal{J} \otimes \mathcal{K})}_{\text{isometry}}$

such that, $\forall X \in L(\mathcal{H})$

$$\Phi(X) = \mathrm{Tr}_{\mathcal{K}}(AXA^*)$$