Optimal lower bound of the average indeterminate length lossless quantum block encoding

George Androulakis

University of South Carolina

ISIT 2024 in Athens Greece

(joint work with R. Wosti)

Outline

1. Special block codes, (adaptive encoding)
2. Constrained special block codes, (non-adaptive encoding)

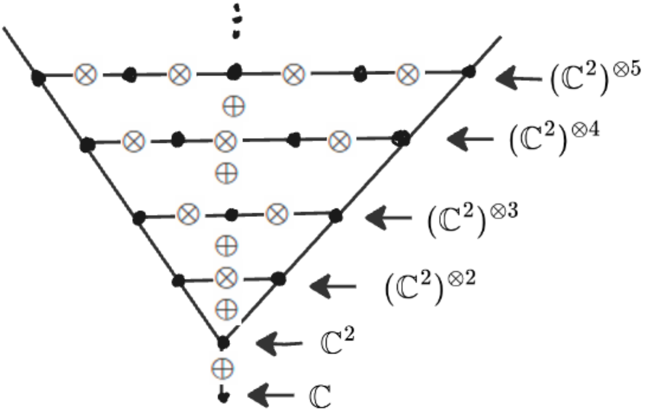
The question:

Consider a general **quantum source** that emits at **discrete** time steps **pure** quantum states which are chosen from a finite alphabet according to some probability distribution which **may depend on the whole history**. Also, fix two positive integers m and l. We encode any tensor product of $\boldsymbol{ml}$ **many states** emitted by the quantum source by breaking the tensor product into $\boldsymbol{m}$ **many blocks** where each block has **length** $\boldsymbol{l}$, and considering sequences of m many isometries so that each isometry encodes one of these blocks into the Fock space, followed by the **concatenation** of their images. We first consider **adaptive encoding** via sequences of isometries that we call **special block codes** in order to ensure that the string of encoded states is uniquely decodable. What is the minimum possible average indeterminate length?

Concatenations in the Fock space

Definition (The Fock space)

$$(\mathbb{C}^2)^{\oplus} = \oplus_{n=0}^{\infty}(\mathbb{C}^2)^{\otimes n}.$$



A picture of the Fock space

Concatenations in the Fock space

Remark

Concatenation *of arbitrary pure quantum states does not always preserve the norm, and hence* **may not be well defined!** *However, the norm is preserved if either or both of the pure quantum states being concatenated are* **length states**. *Warning if both pure quantum states being concatenated are* **indeterminate-length states**, *e.g.*

$$\left\| \frac{1}{\sqrt{2}} (|0\rangle + |00\rangle) \circ \frac{1}{\sqrt{2}} (|0\rangle - |00\rangle) \right\| = \left\| \frac{1}{2} (|00\rangle - |0000\rangle) \right\| \neq 1.$$

[Müller, Rogers, 2008].

Uniquely decodable block quantum codes

Definition (Quantum code)

A **quantum code** *on* $\mathcal{K}$ *is a linear isometry* $U : \mathcal{K} \to (\mathbb{C}^2)^{\oplus}$, *i.e.*

$$U = \sum_{i=1}^{D} |\psi_i\rangle\langle e_i| \,,$$

where $(|\psi_i\rangle)_{i=1}^D$ *is any o.n. sequence and* $(|e_i\rangle)_{i=1}^D$ *is an o.n. basis of* $\mathcal{K}$.

Definition (Concatenation of quantum codes)

$$(U_1 \circ \cdots \circ U_m) |s_1 \otimes \ldots \otimes s_m\rangle := U_1 |s_1\rangle \circ \cdots \circ U_m |s_m\rangle \,, \; |s_1 s_2 \ldots s_m\rangle \in \mathcal{K}^{\otimes m}.$$

Definition (Uniquely decodable block quantum codes)

Let $U_i : \mathcal{K} \to (\mathbb{C}^2)^{\oplus}$ *be quantum codes for* $i = 1, \ldots, m$. $(U_i)_{i=1}^m$ *is called* **uniquely decodable** $\Leftrightarrow U_1 \circ \cdots \circ U_m : \mathcal{K}^{\otimes m} \to (\mathbb{C}^2)^{\oplus}$ *is an isometry.*

Special block codes

Theorem

Let $U^j = \sum_{i=1}^{D} \left|\psi_i^j\middle\rangle\middle\langle e_i^j\right| : \mathcal{K} \to (C^2)^{\oplus}$ *q. codes, for* $j = 1, \ldots, m$. *TFAE:*

- $(U^j)_{j=1}^m$ *is uniquely decodable.*
- $\left\{ \left|\psi_{i_1}^1 \circ \cdots \circ \psi_{i_m}^m\right\rangle : (i_1, \ldots, i_m) \in \{1, \ldots, D\}^m \right\}$ *is an o.n. set.*

Definition (Jointly o.n. sequence)

$(|\psi_i\rangle)_{i=1}^D \subseteq (\mathbb{C}^2)^{\oplus}$ *is called* **jointly orthonormal** *if and only if for every* $m \in \mathbb{N}, \{|\psi_{i_1} \circ \cdots \circ \psi_{i_m}\rangle : (i_1, \ldots, i_m) \in \{1, \ldots, D\}^m\}$ *is an o.n. set.*

Definition (Special block codes, (adaptive encoding))

$$\mathcal{U} = \left\{ U^{n_1, \ldots, n_{(k-1)l}} : 1 \leq k \leq m,\ n_1, \ldots, n_{(k-1)l} \in \{1, \ldots, N\} \right\},$$

$$U^{n_1, \ldots, n_{(k-1)l}} = \sum^{d^l} \left|\psi_i\middle\rangle\middle\langle e_i^{n_1, \ldots, n_{(k-1)l}}\right|, \quad (|\psi_i\rangle)_{i=1}^{d^l} \text{ jointly o.n.}$$

Indeterminate length

Definition (Indeterminate length)

$$\Lambda = \sum_{\ell=0}^{\infty} \ell \Pi_\ell : (\mathbb{C}^2)^{\oplus} \to (\mathbb{C}^2)^{\oplus} \quad \textit{length observable}$$

$$\Pi_\ell : (\mathbb{C}^2)^{\oplus} \to (\mathbb{C}^2)^{\otimes \ell} \quad \textit{orthogonal projection}$$

$$Tr(\rho\Lambda) = \textit{indeterminate length of the state } \rho.$$

Quantum source

Definition (Quantum source and the history probabilities)

$\mathcal{S} = \left\{ \{|s_n\rangle\}_{n=1}^{N} \text{ (**alphabet**)},\ X = (X_n)_{n=1}^{\infty} \right\}$. $d := dim\, Span\, \{|s_n\rangle\}_{n=1}^{N}$.
The **history probabilities***:* $p(n_1, \ldots, n_q) = \mathbb{P}(X_1 = n_1, \ldots, X_q = n_q)$.

Average codeword length of special block codes

Definition (Average codeword length of special block codes)

$$\mathcal{U} = \Big\{ U^{n_1,\dots,n_{(k-1)l}} : 1 \le k \le m, n_1, \dots, n_{(k-1)l} \in \{1, \dots, N\} \Big\},$$

$$\boldsymbol{L}(\mathcal{U}) = \sum_{n_1,\dots,n_{ml}=1}^{N} p(n_1, \dots, n_{ml}) Tr\Bigg($$

$$\Big| U(s_{n_1} \cdots s_{n_l}) \circ U^{n_1,\dots,n_l}(s_{n_{l+1}} \cdots s_{n_{2l}}) \circ \cdots \circ U^{n_1,\dots,n_{(m-1)l}}(s_{n_{(m-1)l+1}} \cdots s_{n_{ml}}) \Big\rangle$$

$$\Big\langle U(s_{n_1} \cdots s_{n_l}) \circ U^{n_1,\dots,n_{n_l}}(s_{n_{l+1}} \cdots s_{n_{2l}}) \circ \cdots \circ U^{n_1,\dots,n_{(m-1)l}}(s_{n_{(m-1)l+1}} \cdots s_{n_{ml}}) \Big|$$

$$\Lambda \Bigg)$$

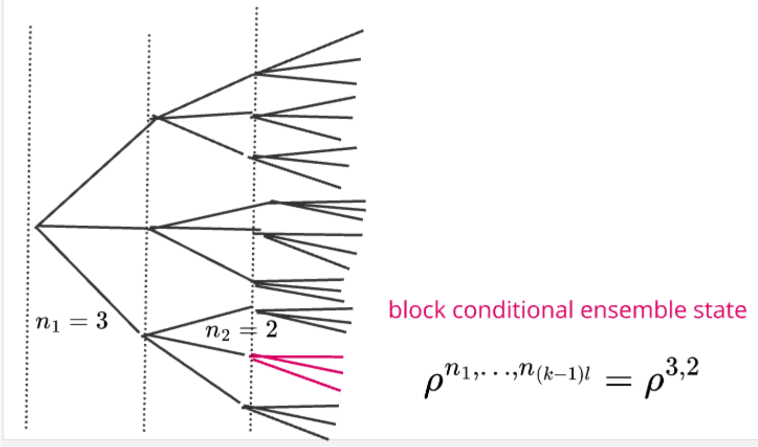
$\boldsymbol{ILS}(\mathcal{S}, \boldsymbol{m}, \boldsymbol{l})$ = **infimum** *of the set containing* $\mathbf{L}(\mathcal{U})$ *for every* **special** *block code* $\mathcal{U}$ *which encodes* m *blocks each of size* l.

Conditional ensemble states

Definition (kth block conditional ensemble state)

$$
\begin{aligned}
&\rho^{n_1,\dots,n_{(k-1)l}} \\
&= \sum_{n_{(k-1)l+1},\dots,n_{kl}=1}^{N} \frac{p(n_1,\dots,n_{kl})}{p(n_1,\dots,n_{(k-1)l})} \left| s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right\rangle\!\left\langle s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right| \\
&= \sum_{n_{(k-1)l+1},\dots,n_{kl}=1}^{N} p(n_{(k-1)l+1},\dots,n_{kl} | n_1,\dots,n_{(k-1)l}) \\
&\qquad \left| s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right\rangle\!\left\langle s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right| .
\end{aligned}
$$

A picture for the kth block conditional ensemble state



The infimum length over all adaptive encodings

Theorem (How to compute $ILS(\mathcal{S}, m, l)$)

$$\rho^{n_1,\ldots,n_{(k-1)l}} = \sum_{i=1}^{d^l} \lambda_i^{n_1,\ldots,n_{(k-1)l}} \left| \lambda_i^{n_1,\ldots,n_{(k-1)l}} \right\rangle\!\left\langle \lambda_i^{n_1,\ldots,n_{(k-1)l}} \right| \quad (sp.\ dec., \searrow).$$

$$\mathfrak{L} = \left\{ (\ell_1, \ldots, \ell_{d^l}) : \ell_i \in \mathbb{N} \cup \{0\}, \ell_1 \leq \ell_2 \leq \cdots \leq \ell_{d^l},\ and\ \sum_{i=1}^{d^l} 2^{-\ell_i} \leq 1 \right\}.$$

$$\mathcal{F}_S((\ell_i)_{i=1}^{d^l}) := \sum_{j=2}^{m} \left(\sum_{n_1,\ldots,n_{(j-1)l}=1}^{N} p(n_1, \ldots, n_{(j-1)l}) \sum_{i=1}^{d^l} \lambda_i^{n_1,\ldots,n_{(j-1)l}} \ell_i \right) + \sum_{i=1}^{d^l} \lambda_i \ell_i, \qquad for\ every\ (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}.$$

Then, $ILS(\mathcal{S}, m, l) = \min\{\mathcal{F}_S((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\}$.

The optimal adaptive encoding

Theorem (The optimal adaptive encoding $\mathcal{V}$)

Assume that $\mathcal{F}_S$ achieves its minimum on $\mathfrak{L}$ at the point $(\ell_i)_{i=1}^{d^l} \in \mathfrak{L}$.
Apply the Kraft-McMillan inequality to $(\ell_i)_{i=1}^{d^l}$ to obtain uniquely decodable $(\omega_i)_{i=1}^{d^l}$.
Then, the corresponding qubit strings $(|\omega_i\rangle)_{i=1}^{d^l} \in (\mathbb{C}^2)^{\oplus}$ form a jointly orthonormal sequence.
For $k \in \{1, \dots, m\}$, and $n_1, \dots, n_{(k-1)l} \in \{1, \dots, N\}$, define $V^{n_1,\dots,n_{(k-1)l}} : \mathcal{H}^{\otimes l} \to (\mathbb{C}^2)^{\oplus}$, by

$$V^{n_1,\dots,n_{(k-1)l}} = \sum_{i=1}^{d^l} \Big|\omega_i\Big\rangle\Big\langle\lambda_i^{n_1,\dots,n_{(k-1)l}}\Big|.$$

$$\mathcal{V} = \Big\{V^{n_1,\dots,n_{(k-1)l}} : k \in \{1, \dots, m\},\ \textit{and}\ n_1, \dots, n_{(k-1)l} \in \{1, \dots, N\}\Big\},$$

$$\min\{\mathcal{F}_S((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\} = L(\mathcal{V}).$$

Constrained special block codes, (non-adaptive encoding)

Definition (Special block codes (adaptive encoding))

$$\mathcal{U} = \Big\{ U^{n_1,\ldots,n_{(k-1)l}} : 1 \leq k \leq m,\ n_1,\ldots,n_{(k-1)l} \in \{1,\ldots,N\} \Big\},$$

$$U^{n_1,\ldots,n_{(k-1)l}} = \sum_{i=1}^{d^l} \Big| \psi_i \Big\rangle \Big\langle e_i^{n_1,\ldots,n_{(k-1)l}} \Big|, \quad (|\psi_i\rangle)_{i=1}^{d^l} \text{ jointly o.n.}$$

Definition (Constrained special block codes, (non-adaptive encoding))

$$\mathcal{U} = \Big\{ U^{n_1,\ldots,n_{(k-1)l}} : 1 \leq k \leq m,\ n_1,\ldots,n_{(k-1)l} \in \{1,\ldots,N\} \Big\},$$

$$U^{n_1,\ldots,n_{(k-1)l}} = U_k = \sum_{i=1}^{d^l} \Big| \psi_i \Big\rangle \Big\langle e_i^{k} \Big|, \quad (|\psi_i\rangle)_{i=1}^{d^l} \text{ jointly o.n.}$$

The question:

Consider a general **quantum source** that emits at **discrete** time steps **pure** quantum states which are chosen from a finite alphabet according to some probability distribution which **may depend on the whole history**. Also, fix two positive integers m and l. We encode any tensor product of **ml many states** emitted by the quantum stochastic source by breaking the tensor product into **m many blocks** where each block has **length l**, and considering sequences of m many isometries so that each isometry encodes one of these blocks into the Fock space, followed by the **concatenation** of their images. We now consider **non-adaptive encoding** via sequences of isometries that we call **constrained special block codes** in order to ensure that the string of encoded states is uniquely decodable. What is the minimum possible average indeterminate length?

Codeword length of constrained special block codes

Definition (Codeword length of constrained special block codes)

$$\mathcal{U} = \Big\{ U_k : 1 \leq k \leq m \Big\},$$

$$\boldsymbol{L}(\mathcal{U}) = \sum_{n_1,\dots,n_{ml}=1}^{N} p(n_1,\dots,n_{ml}) Tr\Bigg(\Big| U_1(s_{n_1}\cdots s_{n_l}) \circ U_2(s_{n_{l+1}}\cdots s_{n_{2l}}) \circ \cdots \circ U_m(s_{n_{(m-1)l+1}}\cdots s_{n_{ml}}) \Big\rangle \Big\langle U_1(s_{n_1}\cdots s_{n_l}) \circ U_2(s_{n_{l+1}}\cdots s_{n_{2l}}) \circ \cdots \circ U_m(s_{n_{(m-1)l+1}}\cdots s_{n_{ml}}) \Big| \Lambda \Bigg)$$

$\boldsymbol{ILC}(\mathcal{S}, \boldsymbol{m}, \boldsymbol{l})$ = **infimum** *of the set containing* $\mathbf{L}(\mathcal{U})$ *for every* **constraint** *special block code* $\mathcal{U}$ *which encodes m blocks each of size l.*

kth block ensemble state

Definition (kth block conditional ensemble state)

$$\rho^{n_1,\ldots,n_{(k-1)l}}$$

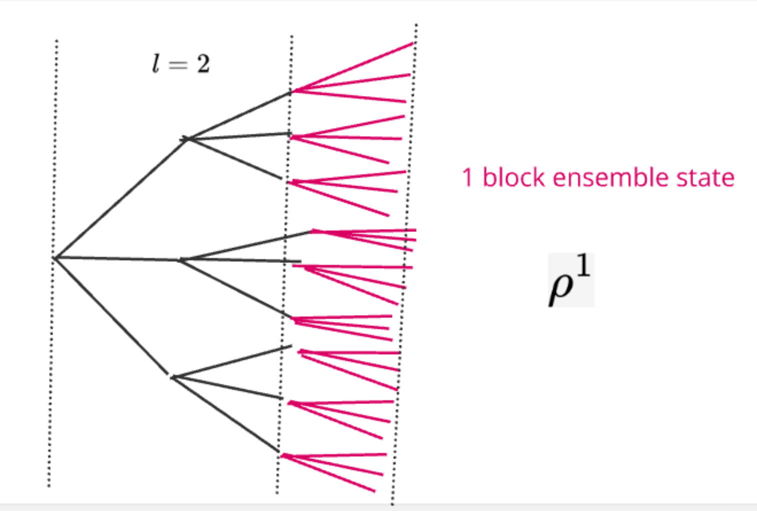
$$= \sum_{n_{(k-1)l+1},\ldots,n_{kl}=1}^{N} \frac{p(n_1,\ldots,n_{kl})}{p(n_1,\ldots,n_{(k-1)l})} \left| s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right\rangle\!\left\langle s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right|$$

$$= \sum_{n_{(k-1)l+1},\ldots,n_{kl}=1}^{N} p(n_{(k-1)l+1},\ldots,n_{kl} | n_1,\ldots,n_{(k-1)l}) \left| s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right\rangle\!\left\langle s_{n_{(k-1)l+1}} \cdots s_{n_{kl}} \right| .$$

Definition (kth block ensemble state)

$$\rho^{k} = \sum_{n_1,\ldots,n_{(k-1)l}=1}^{N} p(n_1,\ldots,n_{(k-1)l}) \rho^{n_1,\ldots,n_{(k-1)l}}$$

A picture for the kth block ensemble state



The infimum length over all non-adaptive encodings

Theorem (How to compute $ILC(\mathcal{S}, m, l)$)

$$\rho^k = \sum_{i=1}^{d^l} \lambda_i^k \left|\lambda_i^k\right\rangle\left\langle\lambda_i^k\right| \quad (\textit{sp. dec.}, \searrow).$$

$$\mathfrak{L} = \left\{ (\ell_1, \ldots, \ell_{d^l}) : \ell_i \in \mathbb{N} \cup \{0\}, \ell_1 \leq \ell_2 \leq \cdots \leq \ell_{d^l}, \textit{ and } \sum_{i=1}^{d^l} 2^{-\ell_i} \leq 1 \right\}.$$

$$\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) := \sum_{k=1}^{m} \sum_{i=1}^{d^l} \lambda_i^k \ell_i \quad \textit{for every } (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}.$$

Then, $ILC(\mathcal{S}, m, l) = \min\{\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\}$.

The optimal non-adaptive encoding

Theorem (The optimal non-adaptive encoding $\mathcal{V}$)

Assume that $\mathcal{F}_C$ achieves its minimum on $\mathfrak{L}$ at the point $(\ell_i)_{i=1}^{d^l} \in \mathfrak{L}$. Apply the Kraft-McMillan inequality to $(\ell_i)_{i=1}^{d^l}$ to obtain uniquely decodable $(\omega_i)_{i=1}^{d^l}$.
Then, the corresponding qubit strings $(|\omega_i\rangle)_{i=1}^{d^l} \in (\mathbb{C}^2)^{\oplus}$ form a jointly orthonormal sequence.
For $k \in \{1, \ldots, m\}$, define $V^k : \mathcal{H}^{\otimes l} \to (\mathbb{C}^2)^{\oplus}$, by

$$V^k = \sum_{i=1}^{d^l} \left|\omega_i\middle\rangle\middle\langle\lambda_i^k\right|.$$

$$\mathcal{V} = \left\{ V^k : k \in \{1, \ldots, m\} \right\}.$$

Then,

$$\min\{\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\} = L(\mathcal{V}).$$

An application for stationary quantum source

Definition

Stationary quantum sources A quantum source $\mathcal{S}$ is **stationary** *(or* **translation-invariant***) if the associated stochastic process X is invariant with respect to the translation map, i.e.*

$$\mathbb{P}(X_1 = n_1, \ldots, X_q = n_q) = \mathbb{P}(X_{k+1} = n_1, \ldots, X_{k+q} = n_q) \tag{1}$$

for every $k \in \mathbb{N}$, $q \in \mathbb{N}$ and $(n_1, \ldots, n_q) \in \{1, \ldots, N\}^q$.

Remark

If $\mathcal{S}$ is a stationary quantum source, with a finite alphabet $(|s_i\rangle)_{i=1}^N$ and history probabilities p, then for every $m, l \in \mathbb{N}$, the kth block conditional ensemble state ρ^k is equal to the **average state after l emissions**

$$\rho_l := \sum_{n_1,\ldots,n_l=1}^{N} p(n_1, \ldots, n_l) \, |s_{n_1} \cdots s_{n_l}\rangle\langle s_{n_1} \cdots s_{n_l}| \, ,$$

The infimum length over all non-adaptive encodings for stationary source

Theorem (How to compute $ILC(\mathcal{S}, m, l)$ for a stationary source)

$$\rho^k = \rho_l = \sum_{i=1}^{d^l} \lambda_i \, |\lambda_i\rangle\langle\lambda_i| \quad (sp.\ dec., \searrow).$$

$$\mathfrak{L} = \left\{ (\ell_1, \ldots, \ell_{d^l}) : \ell_i \in \mathbb{N} \cup \{0\}, \ell_1 \leq \ell_2 \leq \cdots \leq \ell_{d^l},\ and\ \sum_{i=1}^{d^l} 2^{-\ell_i} \leq 1 \right\}.$$

$$\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) = m \sum_{i=1}^{d^l} \lambda_i \ell_i \quad for\ every\ (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}.$$

Then, $ILC(\mathcal{S}, m, l) = \min\{\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\}$.

The optimal non-adaptive encoding for stationary source

Theorem (The optimal non-adaptive encoding for stationary source)

Assume that $\mathcal{F}_C$ achieves its minimum on $\mathfrak{L}$ at the point $(\ell_i)_{i=1}^{d^l} \in \mathfrak{L}$.
Apply the Kraft-McMillan inequality to $(\ell_i)_{i=1}^{d^l}$ to obtain uniquely decodable $(\omega_i)_{i=1}^{d^l}$.
Then, the corresponding qubit strings $(|\omega_i\rangle)_{i=1}^{d^l} \in (\mathbb{C}^2)^{\oplus}$ form a jointly orthonormal sequence.
For $k \in \{1, \ldots, m\}$ define $V^k = V := \mathcal{H}^{\otimes l} \to (\mathbb{C}^2)^{\oplus}$, by

$$V = \sum_{i=1}^{d^l} |\omega_i\rangle\langle\lambda_i| \,.$$ *"Q. Huffman code" [Braunstein, Fuchs, Gottesman, Lo]*

$$\mathcal{V} = \Big\{ V^k : k \in \{1, \ldots, m\} \Big\}.$$

Then,

$$\min\{\mathcal{F}_C((\ell_i)_{i=1}^{d^l}) : (\ell_i)_{i=1}^{d^l} \in \mathfrak{L}\} = L(\mathcal{V}).$$

An application for stationary quantum source

We recover the following result of [Bellomo, Bosyk, Holik, Zozor, (2017)]:

Corollary (Average asymptotic codeword length for stationary source)

Consider a stationary quantum source $\mathcal{S}$ with alphabet $\{|s_n\rangle\}_{n=1}^{N}$ and history probabilities p. For $l \in \mathbb{N}$ consider the average state after l emissions:

$$\rho_l := \sum_{n_1,\ldots,n_l=1}^{N} p(n_1,\ldots,n_l)\,|s_{n_1}\cdots s_{n_l}\rangle\langle s_{n_1}\cdots s_{n_l}|\,.$$

Then,

$$\lim_{l\to\infty}\frac{ILC(\mathcal{S},1,l)}{l} = \lim_{l\to\infty}\frac{\mathcal{S}(\rho_l)}{l}.$$

Thank you!