Just a warning:

Day two
elliptic curves seminar
presenter:
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people.math.sc.edu/alicia/curvesHint.txt

We will now be

defining/explaining

an elliptic curve

as a scheme!

[first we attempt to
sneak in the
Back door]

why: Because I can...

Recall: (from Alicia) ← people.math.sc.edu/alicul/curves17.html

def^n "An elliptic curve is a smooth projective curve of genus 1" → w/ a specified point

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So let's generalize this idea (and get what an elliptic curve "is")

what do we want/need:

want: for the curve to still be defined by a cubic polynomial

like:

$y^2 = x^3 + ax + b$

from last time →

what about need

needed: (our space to be topological!) ← Duh! ← so we can have closed sets...

(a) • our space to have a notion of "projective" (and have a notion of some sort of special point like the are @ infinity)

[* generalized projective space]

(b) • our space to have a notion of "dimension"

(so curves can have dimension 1)

[* krull dimension]

(c) • our curves need a notion of "niceness" or "smoothness"

[* non-singular]

(d) • our curves to have a notion of genus

[* genus from Riemann-Roch]

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so let's just get to it

(a) *projective space: (my favourite interpretation) (generalized)

* make a note that we look at the algebraic closure kinda for the motivation of why we look at genus embedding *

Let k be a field, denote $\bar{k}$ as the algebraic closure

denote $\bar{k}[x_0 - x_n]$ as the polynomials

now denote

$$\text{proj}\ \bar{k}[x_0 - x_n] = \{ p \trianglelefteq \bar{k}[x_0 - x_n] \mid p \text{ homogeneous prime}, p \not\supseteq \langle x_0 - x_n \rangle \}$$

def^n ① homogeneous ideal
② for x, y homogeneous s.t. $x \cdot y \in p$ either $x \in p$ or $y \in p$

def^n S_i homogeneous graded component for $S_i \cap p$ (the i^{th} projection) is in p

* the algebraic closure

We can put the Zariski Topology on this space by defining closed sets

$$V(p) = \{ q \in \text{proj}\ \bar{k}[x_0 - x_n] \mid q \supseteq p \}$$

and open as the complements...

* these are called the (projective) Varieties! *

(a) - [questions may arise as how this notion aligns with the notion shown by Milne / Silverman] - "The Arithmetic of elliptic curves"

people.math.sc.edu/hlicical/seminars/cmss17.html

Taken from Hartshorne "Algebraic geometry"

Recall:

$$Z(T) = \{ p \in \mathbb{P}^n_{\bar{k}} \mid f(p) = 0 \text{ for all } f \in T \}$$

$Y \subset \mathbb{P}^n_{\bar{k}}$ is called algebraic if there exists a set T of homogenous elements of $\bar{k}[x_0 - x_n]$ such that $Y = Z(T)$

Taken from Silverman → and is called a variety when $Y = Z(p)$ for some $p \in \operatorname{proj} \bar{k}[x_0 - x_n]$

"The Arithmetic of Elliptic curves" (i.e. it's generated by a homogenous prime!)

{to see equivalent w/ headstart definition see exercises of Matsumura}

{So this is our (projective) Topological space we will be playing in}

(note nothing in this definition makes use of k being a field or of that we look at it over the algebraic closure we could do the same with a ring and it's internal closure (an integral closure) or whatever, or wherever we want to play, or even that it's polynomials so we can do this w/ any graded ring, also replacing prime with primitive and dropping the commutivity can work or even using non-associative rings ... or we can do anything it's just ducking words on a page!)

The joys of generalization... →

That is, our Topological space is

$$\mathbb{P}^n_{\bar{k}} = \underline{X} = \operatorname{proj} \bar{k}[x_0 - x_n]$$

open sets are compliment of closed sets

closed sets = arbitary intersections of $V(p)$, where $p \in X$

and finite unions

{aside} (if needed)

How does this "align" w/ our previous definition?

Start w/ 1-dimension

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old: $\mathbb{P}'_{\bar{k}} = \{ [x_0 : x_1] : \text{not both zero}, \quad [x_0 : x_1] \sim [y_0 : y_1] \text{ iff } \exists \lambda \in \bar{k}^\times \text{ s.t. } \lambda x_0 = y_0 \text{ and } \lambda x_1 = y_1 \}$

new: $\operatorname{proj} \bar{k}[x_0, x_1] \supseteq \mathbb{P}'_{\bar{k}}$

let $[\alpha_0 : \alpha_1] \in \mathbb{P}'_{\bar{k}}$, WLOG assume $\alpha_0 \neq 0$ then $[\alpha_0 : \alpha_1] \sim \left[1 : \frac{\alpha_1}{\alpha_0}\right]$

then denote

$$\mathcal{P} := \left\langle x_0 - \frac{\alpha_1}{\alpha_0} x_1 \right\rangle \in \operatorname{proj} \bar{k}[x_0, x_1]$$

- irreducible since degree 1 (also gives homogeneity) i.e. prime
- $\mathcal{P} \supseteq \langle x_0, x_1 \rangle$ since $x_1 \notin \mathcal{P}$

* So note in general this is "larger" than $\mathbb{P}'_{\bar{k}}$

So it's not exactly the same

So I'll refer to this as one of the following

- projective scheme space
- scheme's projective space
- theirs projective space
- generalized projective space

* → So this generalizes the idea of projective space *

...still encompasses our old notion *

def^n for a hypersurface the singular points are those @ which all of the partials are zero

our last one...

ex) $y^2 - x^2(x+1) = 0$

$$\frac{\partial c}{\partial y} = 2y \text{, so } \left.\frac{\partial c}{\partial y}\right|_{(0,0)} = 0$$

$$\frac{\partial c}{\partial x} = -2x(x+1) - x^2 \text{, so } \left.\frac{\partial c}{\partial x}\right|_{(0,0)} = 0$$

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how about in (non-graded) projective space...

Q & don't we get (0,0) is not in here?

A just trying to trick you

$$y^2 z - x^3 - x^2 z = 0$$

& check those partials @

$[0:0:1]$

(check homework assignment)

* dimension of a projective variety:

(b) $\dim \frac{k[x_1 - x_n]}{p}$ where dim is the krull dimension...
(generalized idea of generation)

[where $\hat{p}$ is the projection in $\hat{A}^{n}_{k}$]

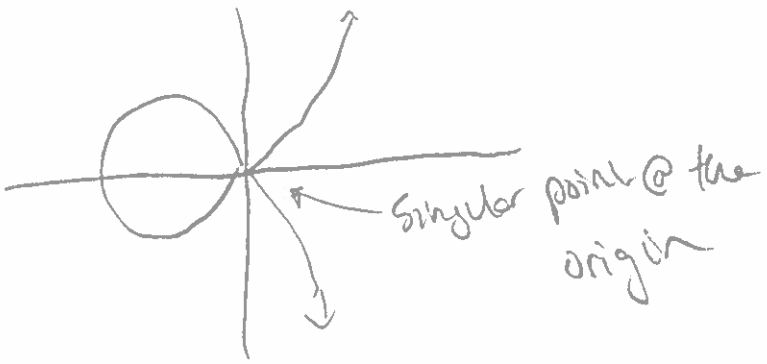
(c) * Smoothness: First what's Not smooth...

a curve is not smooth if
it has singular points

idea
"def^n" [in the ~~geometric~~ analytical sense]

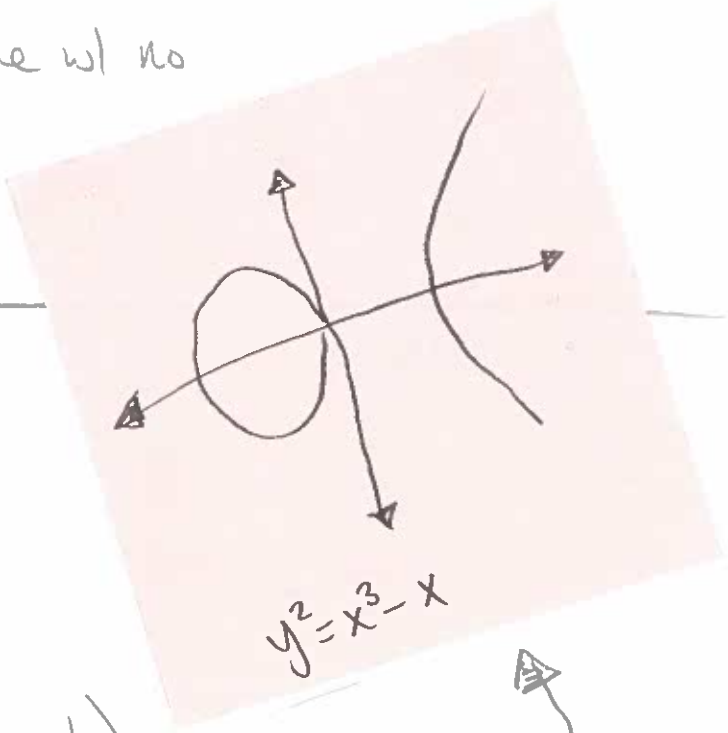
i a point on the curve is called singular
if the tangent space @ that point can not
be "regularly" defined

ex $y^2 - x^2(x+1) = 0$



So... def^n

a smooth curve is a curve w/ no singular points



(1) * genus

arithmetic genus: ($\dim C$)

def^n $P_a(C) = (-1)^r (c(0) - 1)$

↑ the "nonsingular version"

ex Let's check dimension, genus, and smoothness

$y^2 = x^3 - x$

dimension: $\dim \frac{\mathbb{C}[x,y]}{\langle y^2 - x^3 - x \rangle} = 1$ {Krull dim}

Because $\dim \mathbb{C}[x,y] = 2$ and $\langle y^2 - x^3 - x \rangle$ prime

genus: $P_a(C) = (-1)^1 (0 - 1) = 1$

$\frac{\partial C}{\partial y} = 2yz$, $\frac{\partial C}{\partial x} = 3x - z$, $\frac{\partial C}{\partial z} = y^2 - x$

only vanishes @ $(0,0,0) \notin \mathbb{P}^2$