*§0: Notation & References* ..........

Throughout these notes, I attempted to use notation consistent with Perrin's notes. We let $k$ denote an algebraically closed field of arbitrary characteristic, although all of the examples will take place in characteristic zero. We will take a *variety* to be a reduced separated scheme of finite type over $k$.

The primary reference for these notes is [4]. (And [3] as a supplement) For facts concerning Lie groups and Lie algebras, see [1], and for general algebraic geometry facts, [2] was used.

*§1: What is an Algebraic Group?* ..........

**Definition 1.** *An* algebraic group *is a variety $G$ which is also a group. We require also that the maps defining the group structure of $G$:*

- *(Multiplication)* $\mu : G \times G \to G$ *with* $\mu(x, y) = xy$
- *(Inverses)* $i : G \times G \to G$ *with* $i(x) = x^{-1}$
- *(Identity)* $e_G : \mathrm{Spec}(k) \to G$ *with image* $e_G$ *of* $G$

*are regular maps on the variety $G$.*

Now that we have a proper definition for an *algebraic group*; we assume that $G$ is an algebraic group throughout these notes unless stated otherwise.

We also have the following definitions / facts:

- We call $G$ a *Linear Algebraic Group* if $G$ is an affine variety.
- A connected [1] algebraic group which is also complete is called and *abelian variety.*
- If $\varphi : G \to G'$ is a morphism of varieties between two algebraic groups $G, G'$ such that $\varphi$ is a group homomorphism, then we say that $\varphi$ is a *homomorphism of algebraic groups.*
- A *closed subgroup* $H$ of an algebraic group $G$ is a closed subvariety of $G$ which is also a subgroup of $G$.
- Given two algebraic groups $G, G'$, the product $G \times G'$ with the group structure of a direct product is an algebraic group.

**Definition 2.** *If we assume that $G = \mathrm{Spec}\, A$ for a finitely generated algebra $A$, (ie. $G$ is a linear algebraic group with $k[G] = A$) the morphisms $\mu, i, e_G$ defined above induce the following algebra homomorphisms:*

- $\mu$ *induces* $\Delta : k[G] \to k[G] \otimes k[G] = k[G \times G]$ *(comultiplication)*

[1] By 'connected' I mean 'connected in the Zariski topology'. Perhaps this is obvious, but a connected algebraic group need not be connected as a Lie group. Take for example $\mathbb{R}^{\times}$, which is a connected algebraic group but is *not* connected as a Lie group.

- *$i$ induces $\imath : k[G] \to k[G]$ (antipode)*
- *$e_G$ induces $\epsilon : k[G] \to k$ (coidentity)*

*A $k$-algebra $A$ with morphisms defined as above is called a* Hopf Algebra*, which we will denote as $k[G]$.*

Recall that associated to an affine variety $X$, we have the *coordinate ring* of $X$- the global sections of the structure sheaf of $X$. The *Hopf algebra* associated to an algebraic group $G$ is much like the coordinate ring of an affine variety, but also encodes information about the group structure of $G$.

**Examples.**

- $G = \mathbb{A}^1_k \cong k$ (The *additive group*, denoted $\mathbb{G}_a$)

  In this case, $k[G] = k[T]$ for some variable $T$. The Hopf algebra maps are the following:

  - $\Delta : k[T] \to k[T] \otimes k[T]$ given by $\Delta(T) = T \otimes 1 + 1 \otimes T$ (comultiplication)
  - $\imath : k[T] \to k[T]$ given by $\imath(T) = -T$ (antipode)
  - $\epsilon : k[T] \to k$ given by $\epsilon(T) = 0$ (coidentity)

- $G = \mathbb{A}^1 \setminus \{0\} \cong k^\times$ (The *multiplicative group*, denoted $\mathbb{G}_m$) This is $\mathrm{GL}_1$.

  Here, $k[G] = k[T^{\pm 1}]$. The Hopf algebra maps are given by the following:

  - $\Delta(T) = T \otimes T$
  - $\imath(T) = T^{-1}$
  - $\epsilon(T) = 1$

- $G = \mathrm{GL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$.

  Here, $G$ is a linear algebraic group. Notice that $G = \operatorname{Spec} A$, where $A = \frac{k[a,b,c,d,x]}{((ad-bc)\cdot x - 1)}$.

- Any closed (under the Zariski topology) subgroup of $\mathrm{GL}_n$

*§2: Basic Properties of Algebraic Groups* ........................................

**Theorem 1.** (Chevalley) *Let $G$ be an algebraic group. There exists a maximal linear algebraic subgroup $G_{aff}$ of $G$; this subgroup is normal and the quotient $G/G_{aff}$ is an abelian variety.*

**Proposition 1.** *Let $G$ an algebraic group.*

1. *There exists a unique irreducible component $G^0$ of $G$ containing the identity $e_G$. It is a closed normal subgroup of $G$, and has finite index. We call this the* identity component *of $G$.*
2. *The subgroup $G^0$ is the unique connected component containing $e_G$. The connected components and the irreducible components of $G$ coincide.*

*3. Any closed subgroup of $G$ with finite index contains $G^0$.*

**Lemma 1.** *Let $U$, $V$ be dense open subsets of $G$. Then, $UV = G$.*

**Lemma 2.** *Let $H$ be a subgroup of $G$.*

- *The closure $\overline{H}$ of $H$ is a subgroup of $G$.*
- *If $H$ contains a non-empty open subset of $\overline{H}$, then $H$ is closed.*

**Proposition 2.** *Let $\phi : G \to G'$ a morphism of algebraic groups.*

- *The kernel* $\ker \phi$ *is a closed normal subgroup.*
- *The image $\phi(G)$ is a closed subgroup of $G$.*
- $\phi(G^0) = \phi(G)^0$.

**Definition 3.** *Let $H$ and $K$ be subgroups of $G$. The subgroup generated by the commutators, ie. elements of the form $hkh^{-1}k^{-1}$ is denoted by $(H, K)$.*

**Proposition 3.** *If $H$ and $K$ are closed subgroups of $G$ and one of $H$, $K$ is connected, then $(H, K)$ is closed and connected.*

**Examples.** A few important examples of closed subgroups.

- Upper triangular matrices in $\mathrm{GL}_n$.
- Diagonal matrices in $\mathrm{GL}_n$.

*§3: Actions on Varieties* ..........................................................................

**Definition 4.** *Let $X$ be a variety with an action of an algebraic group $G$.*

- *Let $a_X : G \times X \to X$ with $a_X(g, x) = g \cdot x$ be the map given by the action of $G$ on $X$. We say that $X$ is a $G$-*variety*, or a $G$-*space *if $a_X$ is a morphism.*
- *A $G$-space with a transitive action (ie. for any $x, y \in X$ there exists $g \in G$ so that $g \cdot x = y$) of $G$ is called a* homogeneous space.
- *A morphism $\phi : X \to Y$ between $G$-spaces is called* equivariant *if the following diagram commutes:*

$$\begin{tikzcd} G \times X \arrow[r, "a_X"] \arrow[d, "\mathrm{Id} \times \phi"'] & X \arrow[d, "\phi"] \\ G \times Y \arrow[r, "a_Y"'] & Y \end{tikzcd}$$

- *Let $X$ be a $G$-space and $x \in X$. The* orbit *of $x \in X$ is the image $G \cdot x = a_X(G \times \{x\})$. The* isotropy group *of $x \in X$ (or* stabilizer *of $x$) is the subgroup $G_x = \{g \in G \mid g \cdot x = x\}$.*
- *If $X$ is a homogeneous space for the action of $G$, and all of the isotropy groups are trivial, then we say that $X$ is a* principal homogeneous space *or a* torsor*.*

**Examples.**

- The group $G$ itself can be viewed as a $G$-space. To see this, let $a_G : G \times G \to G$ be given by $(g, h) \mapsto ghg^{-1}$. In this case, the orbits of the action are conjugacy classes, and the isotropy subgroups are centralizers.
- $G$ can also act on itself via translation. Define $a_G : G \times G \to G$ by $(g, h) \mapsto gh$ (or $(g, h) \mapsto hg$). In this case, the action is transitive.
- Let $V$ a finite dimensional vector space. Define $a_V : \mathrm{GL}(V) \times V \to V$ to be given by $(f, v) \mapsto f(v)$. This gives a $\mathrm{GL}(V)$-space structure on $V$.

**Lemma 3.** *Let $X$ be a $G$-space.*

- *Any orbit is open in its closure.*
- *There is at least one closed orbit in $X$.*

As expected, a $G$-space $X$ with $X$ affine descends to a map of algebras. Write $X = \mathrm{Spec}(k[X])$. The action $a_X : G \times X \to X$ is given by a map $a_X^{\#} : k[X] \to k[G] \otimes k[X]$. We define a representation of abstract groups $r : G \to \mathrm{GL}(k[X])$ defined by $(r(g)f)(x) = f(g^{-1}x)$. An element $g \in G$ gives a map $ev_g : k[G] \to k$, and we get the following composition:

$$r(g) : k[X] \xrightarrow{a_X^{\#}} k[G] \otimes k[X] \xrightarrow{ev_{g^{-1}}} k \otimes k[X] = k[X].$$

**Proposition 4.** *Let $V$ a finite dimensional subspace of $k[X]$.*

- *There is a finite dimensional subspace $W$ of $k[X]$ so that $V \subset W$ and $W$ is stable under the action of $r(g)$ for all $g \in G$.*
- *The subspace $V$ is stable under the action of $r(g)$ for all $g \in G$ if and only if $a_X^{\#}(V) \subset k[G] \otimes V$. If this is the case, then $r_V : G \times V \to V$ given by $(g, f) \mapsto (ev_g \otimes \mathrm{Id}) \circ a_X^{\#}(f)$ is a rational representation.*

Of particular importance to us will be the action of $G$ on $k[G]$ via left and right translation.

**Notation.** For $g \in G$, we define $\lambda$ and $\rho$ as follows:

$$\lambda(g) : k[G] \to k[G] \qquad \rho(g) : k[G] \to k[G].$$

These are just the the representations of $G$ in $\mathrm{GL}\left(k[G]\right)$ induced by left and right translation respectively. More explicitly, for $g \in G$ and $f \in k[G]$, we have:

$$(\lambda(g)f)(x) = f\left(g^{-1}x\right) \qquad \text{and} \qquad (\rho(g)f)(x) = f(xg),$$

for all $x \in G$. [2] Notice that $\lambda$ and $\rho$ are faithful representations.

**Theorem 2.** *(!!) Any linear algebraic group is a closed subgroup of* $\mathrm{GL}_n$ *for some* $n$*.*

*§4: Derivations & Tangent Spaces* . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Definition 5.** *Let $R$ a commutative ring, $A$ and $R$-algebra, and $M$ and $A$-module. An* R-derivation *of $A$ in $M$ is a linear map $D : A \to M$ such that for all $a, b \in A$, we have the following:*

$$D(ab) = aD(b) + D(a)b$$

*We denote the set of all such derivations by $\mathrm{Der}_R(A, M)$. Notice that $\mathrm{Der}_R(A, M)$ is an $A$-module; in particular, if $D, D'$ are derivations, then so is $D+D'$. Similarly, for any $a \in A$ and any derivation $D$, we have that $aD$ is a derivation.*

**Proposition 5.** *Let $\phi : A \to B$ a morphism of $R$-algebras, and $\varphi : M \to N$ a morphism of modules.*

- *The map $\mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$ given by $D \mapsto D \circ \phi$ is well defined. It is a morphism of $A$-modules, and its kernel is $\mathrm{Der}_A(B, M)$.*
- *The map $\mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A, N)$ given by $D \mapsto \varphi \circ D$ is well defined and a morphism of $A$-modules.*
- *Given a multiplicative subset $S$ of $A$, and $M$ an $S^{-1}A$-module, there is a natural isomorphism $\mathrm{Der}_R\left(S^{-1}A, N\right) \to \mathrm{Der}_R(A, N)$.*
- *Let $A_1, A_2$ be two $R$-algebras. Write $A = A_1 \otimes_R A_2$ and $M$ an $A$-module. Then, we have that*

$$\mathrm{Der}_R(A, M) \cong \mathrm{Der}_R(A_1, M) \oplus \mathrm{Der}_R(A_2, M).$$

**Definition 6.** *Let $X$ an algebraic variety and $x \in X$. The* tangent space *of $X$ at $x$ is the vector space $\mathrm{Der}_k\left(\mathcal{O}_{X,x}, k(x)\right)$, where $k(x) := \mathcal{O}_{X,x}/\mathfrak{M}_{X,x}$.* [3] *We write this as $T_xX$.*

We have the following facts concerning derivations and tangent spaces.

- When $X$ is an affine variety, we have that $T_xX = \mathrm{Der}_k\left(k[X], k(x)\right)$.
- Let $x \in X$ and $U \subset X$ and open subvariety with $x \in U$. Then, we have that $T_xU = T_xX$. (Since $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$)

[2] These are both *left* actions, despite their names.

[3] Notational remark: Given an algebraic variety $X$ and $x \in X$, we know that $\mathcal{O}_{X,x}$ is a local ring. The notation $\mathfrak{M}_{X,x}$ is just the (unique) maximal ideal of $\mathcal{O}_{X,x}$.

- We have the following isomorphism: $T_xX \cong \left(\mathfrak{M}_{X,x}/\mathfrak{M}_{X,x}^2\right)^\vee$.

**Proposition 6.** *Let $\phi : X \to Y$ a morphism of varieties.*

- *There exists a linear map $d_x\phi : T_xX \to T_{f(x)}Y$. This is the* differential *of $\phi$ at $x$.*
- *Let $\varphi : Y \to Z$ a morphism of varieties. Then, we have the following:*

$$d_x(\varphi \circ \phi) = d_{f(x)}\varphi \circ d_x\phi.$$

- *If $\phi$ is an isomorphism or the identity, then so is $d_x\phi$.*
- *If $\phi$ is a constant map, then $d_x\phi = 0$ for any $x \in X$.*

**Definition 7.** *The* cotangent space *of $X$ at $x$ is $\mathfrak{M}_{X,x}/\mathfrak{M}_{X,x}^2$. From the fact above, we see that it is isomorphic to $(T_xX)^\vee$.*

**Lemma 4.** *Let $\phi : X \to Y$ a closed immersion, then $d_x\phi$ is injective for any $x \in X$. Thus we are able to identify the tangent space $T_xX$ with a subspace of $T_{\phi(x)}Y$.*

**Proposition 7.** *Let $X$ a closed subvariety of $k^n$ and let $I$ be the defining ideal of $X$. Assume also that $I$ is generated by elements $f_1, \ldots, f_r$. Then, for all $x \in X$ we have the following:*

$$T_xX \quad = \quad \bigcap_{k=1}^{r} \ker d_x f_k \quad = \quad \left\{ v \in k^n \mid \sum_{i=1}^{n} v_i \frac{\partial f_k}{\partial x_i}(x) = 0 \text{ for all } k \in [1, r] \right\}$$

**Proposition 8.** *Let $\phi : X \times Y \to Z$ a morphism of varieties, and let $x \in X$ and $y \in Y$. Then, we have the isomorphism: $T_{(x,y)}X \times Y \cong T_xX \oplus T_yY$. Further, modulo this identification we have the following equality:*

$$d_{(x,y)}X\phi = d_x\phi_y + d_y\phi_x$$

*where $\phi_x : Y \to Z$ is given by $\phi_x(y) = \phi(x, y)$ and $\phi_y : X \to Z$ is given by $\phi_y(x) = \phi(x, y)$.*

**Definition 8.** *Let $X$ a variety with $x \in X$, and $n \in \mathbb{Z}_{\geq 0}$. Notice that we can identify $\mathfrak{M}_{X,x}^\vee$ with:*

$$\mathfrak{M}_{X,x}^\vee \cong \{\phi \in \mathcal{O}_{X,x}^\vee \mid \phi(1) = 0\}.$$

*We define the following vector spaces:*

$$\mathrm{Dist}_n(X, x) = \{\phi \in \mathcal{O}_{X,x}^\vee \mid \phi\left(\mathfrak{M}_{X,x}^{n+1}\right) = 0\} \cong \left(\mathcal{O}_{X,x}/\mathfrak{M}_{X,x}^{n+1}\right)^\vee$$

$$\mathrm{Dist}_n^+(X, x) = \{\phi \in \mathrm{Dist}_n(X, x) \mid \phi(1) = 0\} \cong \left(\mathfrak{M}_{X,x}/\mathfrak{M}_{X,x}^{n+1}\right)^\vee$$

*Using this, we set:*

$$\mathrm{Dist}(X, x) = \bigcup_n \mathrm{Dist}_n(X, x) \quad \text{and} \quad \mathrm{Dist}^+(X, x) = \bigcup_n \mathrm{Dist}_n^+(X, x).$$

*The elements of $\mathrm{Dist}(X, x)$ are called the* distributions of $X$ with support in $x$. *We have the identification $\mathrm{Dist}_1^+(X, x) = T_xX$. Distributions provide an algebraic analogue of higher order differential operators on a differential manifold.*

*§5: The Lie Algebra of an Algebraic Group* ........................................

**Definition 9.** *A* Lie algebra $\mathfrak{g}$ *is a vector space together with a bilinear map* $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(a* Lie bracket*) satisfying the following properties:*

- $[x, x] = 0$
- *(Jacobi Identity)* $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ *for all* $x, y, z \in \mathfrak{g}$.

Less precisely, given an algebraic group $G$, I think of the Lie algebra associated to $G$ as being a vector space that is equal to the tangent space at $e_G$.

**Definition 10.** *We have the following related definitions:*

1. *A* morphism of Lie algebras *is a linear map* $\phi : \mathfrak{g} \to \mathfrak{g}'$ *such that* $\phi([x, y]) = [\phi(x), \phi(y)]$ *for all* $x, y \in \mathfrak{g}$. *(In other words, the brackets on* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *play along nicely with* $\phi$*.)*
2. *A* representation *of a Lie algebra* $\mathfrak{g}$ *in a vector space* $V$ *is a morphism of Lie algebras* $\mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}_k(V)$ *where* $\mathfrak{gl}(V)$ *has the Lie structure associated to the commutators.*

**Examples.**

1. Consider $G = \mathrm{GL}_n$. We view $G$ as a subset of $M_n$; ($n \times n$ matrices) in particular it is the complement of $\det = 0$. $G$ is an open subset of $M_n$, and thus the tangent space is all of $M_n$. In this case, the Lie bracket is $[A, B] := AB - BA$.
2. If $A$ is an associative algebra and
$$\mathrm{Der}_k(A) = \{D \in \mathrm{End}_k(A) \mid D(ab) = aD(b) + D(a)b\},$$
then $\mathrm{Der}_k(A)$, together with the bracket $[D, D'] = D \circ D' - D' \circ D$ is a Lie algebra.

**Proposition 9.** *The left and right actions of* $G$ *on* $\mathfrak{gl}(k[G])$ *preserve the subspace of derivations. Further, the subspace* $\mathrm{Der}_k(k[G])^{\lambda(G)}$ *of invariant derivations for the left action is a lie subalgebra of* $\mathrm{Der}_k(k[G])$.

**Definition 11.** *The* Lie algebra $L(G)$ *of the group* $G$ *is* $\mathrm{Der}_k(k[G])^{\lambda(G)}$.

We have the following facts:

- If $H$ is a closed algebraic subgroup of $G$, then $L(H)$ is a Lie subalgebra of $L(G)$.
- The tangent space $T_{e_G}G$ is endowed with a Lie algebra structure which comes from the Lie algebra structure on $L(G)$.
- There is a natural Lie algebra structure on $\mathrm{Dist}(G)$. The bracket is given by $[\eta, \xi] = \eta\xi - \xi\eta$.
- The subspace $\mathrm{Dist}_1^+(G)$ is stable under the Lie bracket given in the previous fact, so it is a Lie subalgebra of $\mathrm{Dist}(G)$. One can identify $\mathrm{Dist}_1^+(G)$ with $T_{e_G}(G)$; the Lie algebra structures on these spaces agree.

- $\mathrm{Dist}(G) = \mathrm{Dist}(G^0)$, and $L(G) = L(G^0)$.

**Definition 12.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. We define* the universal enveloping algebra $U(\mathfrak{g})$ *of $\mathfrak{g}$ as the following quotient:*

$$U(\mathfrak{g}) = T(\mathfrak{g})/\left(x \otimes y - y \otimes x - [x,y]\right),$$

*for all $x, y \in \mathfrak{g}$. Here, $T(\mathfrak{g})$ denotes the tensor algebra constructed from $\mathfrak{g}$.*

The *universal enveloping algebra* of a Lie group $\mathfrak{g}$ can be though of as being the most general associative algebra containing all representations of $\mathfrak{g}$. To clarify this, we have the following proposition:

**Proposition 10.** *Let $\tau : \mathfrak{g} \to U(\mathfrak{g})$ be the natural map.*

- *Let $A$ an associative algebra and let $\phi : \mathfrak{g} \to A$ be a Lie algebra morphism, where the Lie bracket on $A$ is $[a,b] = ab - ba$. Then, there exists a unique algebra morphism $\Phi : U(\mathfrak{g}) \to A$ such that $\phi = \Phi \circ \tau$.*

- *There is an equivalence of categories between* $\mathrm{Rep}(\mathfrak{g})$*, the category of Lie algebra representations of $\mathfrak{g}$, and* $\mathrm{Mod}(U(\mathfrak{g}))$*, the category of $U(\mathfrak{g})$-modules.*

**Theorem 3.** (Poincare-Birkhoff-Witt) *Given a basis of the Lie algebra $\mathfrak{g}$, a basis for the universal enveloping algebra $U(\mathfrak{g})$ can be created.*

*§6: Semisimple & Unipotent Elements* ........................................................

Note: In this section, you need to be careful about the case $\mathrm{char}(k) = p$.

**Definition 13.** *Let $V$ be a vector space. We recall some definitions from linear algebra:*

1. *Any $x \in \mathrm{End}(V)$ which is diagonalisable we will call* semisimple*. Equivalently, if the dimension of $V$ is finite, the minimal polynomial is separable.*

2. *Any $x \in \mathrm{End}(V)$ such that $x^n = 0$ for some $n \in \mathbb{Z}$ is called* nilpotent*. When $x - \mathrm{Id}$ is nilpotent, we say $x$ is* unipotent*.*

3. *Any $x \in \mathrm{End}(V)$ such that for all $v \in V$, $\mathrm{span}\{x^n(v) \mid n \in \mathbb{N}\}$ has finite dimension is called* locally finite*.*

4. *Any $x \in \mathrm{End}(V)$ such that for all $v \in V$ there exists $n \in \mathbb{Z}$ so that $x^n(v) = 0$ we will call* locally nilpotent*. Similarly, for $x \in \mathrm{End}(V)$ such that $\mathrm{Id} - x$ is locally nilpotent, we will call* locally unipotent*.*

**Theorem 4.** (Additive Jordan Decomposition) *Let $x \in \mathfrak{gl}(V)$ locally finite. Let $x \in \mathfrak{gl}(V)$ be locally finite.*

1. *There exists a unique decomposition $x = x_s + x_n$ in $\mathfrak{gl}(V)$ such that $x_s$ is semisimple, $x_n$ is nilpotent, and $x_s$ commutes with $x_n$.*

2. *There exists polynomials $p$ and $q$ in $k[T]$ such that $x_s = p(x)$ and $x_n = q(x)$. In particular, $x_s$ and $x_n$ commute with any endomorphism commuting with $x$.*

3. *If $U \subset W \subset V$ are subspaces such that $x(W) \subset U$, then $x_s$ and $x_n$ also map $W$ into $U$.*

4. *If $x(W) \subset W$, then we have the following equalities: $(x\mid_W)_s = (x_s) \mid W$, $(x\mid_W)_n = (x_n)\mid_W$, $(x\mid_{V/W})_s = (x_s)\mid_{V/W}$, $(x\mid_{V/W})_n = (x_n)\mid_{V/W}$.*

*Note:* We also have a **Multiplicative Jordan Decomposition**. In this case, we have the decomposition $x = x_s x_u$; $x_n$ in the above Theorem is replaced with $x_u$, the unipotent part of $x$.

**Definition 14.** *The elements $x_s$ are called the* semisimple *part of $x \in \mathrm{End}(V)$. Similarly, the elements $x_n$ are called the* nilpotent *part of $x \in \mathrm{End}(V)$. The decomposition $x = x_s + x_n$ is called the* Jordan-Chevalley decomposition.

**Definition 15.**

- *Let $g \in G$. We call $g$* semisimple *if $g = g_s$, and $g$ is* unipotent *if $g = g_u$.*
- *Let $\eta \in \mathfrak{g}$. We say that $\eta$ is* semisimple *if $\eta = \eta_s$, and* nilpotent *if $\eta = \eta_n$.*
- *We denote the semisimple elements of $G$ as $G_s$, and the unipotent elements in $G$ by $G_u$. Similarly, we denote the semisimple elements of $\mathfrak{g}$ as $\mathfrak{g}_s$ and the nilpotent elements of $\mathfrak{g}$ as $\mathfrak{g}_n$.*

**Definition 16.** *Let $G$ an algebraic group.*

1. *We say that $G$ is unipotent if $G = G_u$.*
2. *We say that $G$ is* diagonalizable *if there exists a faithful representation $G \to \mathrm{GL}(V)$ such that the image of $G$ is contained in the subgroup of diagonal matrices.*

**Proposition 11.** *The following statements are equivalent.*

1. *$G$ is diagonalizable.*
2. *$G$ is a closed subgroup of $\mathbb{G}_m^n$.*
3. *$G$ is commutative, and every element of $G$ is semisimple.*

To conclude the talk, we briefly discuss results concerning the structure of commutative groups, and a classification of algebraic groups of dimension one.

**Theorem 5.** *(A structure theorem) Let $G$ be a commutative group, and $\mathfrak{g}$ its Lie algebra.*

- *$G_s$ and $G_u$ are closed subgroups of $G$, and are connected if $G$ is connected. Additionally, the map $G_s \times G_u \to G$ given by $(x, y) \mapsto xy$ is an isomorphism. (The inverse is given by the Jordan decomposition)*
- *$L(G_s) = \mathfrak{g}_s$, $L(G_u) = \mathfrak{g}_u$, and $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_u$.*

**Theorem 6.** *(A classification theorem) Let $G$ be a connected algebraic group of dimension 1. Then, we have that $G = \mathbb{G}_m$ or $G = \mathbb{G}_a$.*

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