Math 122

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Calculus for Business Administration and Social Sciences
5.2: The Definite Integral

5.5: The Fundamental Theorem of Calculus
1 5.2: THE DEFINITE INTEGRAL

2 5.5: THE FUNDAMENTAL THEOREM OF CALCULUS
In the last section, we saw that for a continuous function $f$ on an interval $[a, b]$, the error for Left-Hand Sums and Right-Hand Sums goes to zero as the number of points in the partition becomes large.
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As the error term goes to zero, the Left-Hand Sum increases towards a fixed value and the Right-Hand Sum decreases towards that same value.

The common value that these sums approach is called a *limit*, and we call this particular limit the Definite Integral.
DEFINITION 1

Assume that $f$ is continuous on the interval $[a, b]$. The definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=0}^{n} f(t_i) \Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \Delta t,$$

where the set of $t$-values

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

is a partition of $[a, b]$ into $n$ intervals of length

$$\Delta t = \frac{b - a}{n}.$$
Compute $\int_{x_0}^{x_1} b \, dx$ for $0 < b$. 
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$$\int_{x_0}^{x_1} b \, dx = b(x_1 - x_0).$$
Let \( f(x) = mx + b \) for \( 0 < b, 0 < m \). Compute \( \int_{x_0}^{x_1} f(x) \, dx \) for \( -\frac{b}{m} < x_0 \).
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$$\int_{x_0}^{x_1} f(x) \, dx = f(x_0) (x_1 - x_0) + \frac{1}{2} \left[ f(x_1) - f(x_0) \right] (x_1 - x_0).$$
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\int_{x_0}^{x_1} f(x) \, dx = f(x_0)(x_1 - x_0) + \frac{1}{2} [f(x_1) - f(x_0)] (x_1 - x_0).
\]
Compute $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$. 

Observe $x^2 + y^2 = 1$ is a circle of radius 1 centered at $(0, 0)$ and the area of a circle of radius $r$ is $\pi \cdot r^2$. 

The curve $y = \sqrt{1 - x^2}$ is the top half of this circle, and the integral is the area bounded by this semicircle: 

Therefore $\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{1}{2} \pi (1)^2 = \frac{\pi}{2}$. 

Compute $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$.

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\]
The area under the parabola

is \( \frac{4}{3} rh \). Use this to compute \( \int_{0}^{3} x^2 \, dx \).
The integral is just the area under the parabola:
If we flip the picture upside down, we have the picture

And we note that the red and blue areas are, by symmetry, the same.
GEOMETRIC EXAMPLES

Hence we can compute the area using
Hence we can compute the area using

\[
\text{area} \left( \begin{array}{c}
\end{array} \right) =
\]

\[
\int_3^0 x^2 \, dx = \frac{1}{2} \left[ 6 \cdot 9 - 4 \cdot 9 \cdot 3 \right] = \frac{9}{2}.
\]
Hence we can compute the area using

\[
\text{area } (\text{\textbullet}) = \text{area } (\text{\square}) - \text{area } (\text{\triangle}) - \text{area } (\text{\triangleup})
\]
Hence we can compute the area using

\[
\text{area} \left( \square \right) = \text{area} \left( \square \right) - \text{area} \left( \triangle \right) - \text{area} \left( \triangle \right)
\]

\[
= \text{area} \left( \square \right) - \text{area} \left( \triangle \right) - \text{area} \left( \triangle \right)
\]
Hence we can compute the area using

\[
\text{area } \begin{array}{c}
  \text{\textbullet} \\
  \text{\textunderscore}
\end{array}
\quad = \quad \text{area } \begin{array}{c}
  \text{□}
\end{array}
\quad - \quad \text{area } \begin{array}{c}
  \text{\textbullet}
\end{array}
\quad - \quad \text{area } \begin{array}{c}
  \text{△}
\end{array}
\]

\[
\quad = \quad \text{area } \begin{array}{c}
  \text{□}
\end{array}
\quad - \quad \text{area } \begin{array}{c}
  \text{\textbullet}
\end{array}
\quad - \quad \text{area } \begin{array}{c}
  \text{△}
\end{array}
\]

\[
\Rightarrow 2 \cdot \text{area } \begin{array}{c}
  \text{\textbullet}
\end{array}
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\end{array}
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Hence we can compute the area using
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\]
Therefore
\[
\int_{0}^{3} x^2 \, dx = \]

Hence we can compute the area using

\[
\text{area} (\triangle) = \text{area} (\square) - \text{area} (\triangle) - \text{area} (\triangle)
\]

\[
= \text{area} (\square) - \text{area} (\triangle) - \text{area} (\triangle)
\]

\[\Rightarrow 2 \cdot \text{area} (\triangle) = \text{area} (\square) - \text{area} (\triangle) .\]

Therefore

\[
\int_{0}^{3} x^2 \, dx = \frac{1}{2} \left[ 6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3 \right]
\]
Hence we can compute the area using

\[
\text{area} \left( \frac{\square}{\triangle} \right) = \text{area} \left( \square \right) - \text{area} \left( \triangle \right) - \text{area} \left( \frac{\triangle}{\triangle} \right)
\]

\[
= \text{area} \left( \square \right) - \text{area} \left( \triangle \right) - \text{area} \left( \frac{\triangle}{\triangle} \right)
\]

\[
\Rightarrow 2 \cdot \text{area} \left( \frac{\square}{\triangle} \right) = \text{area} \left( \square \right) - \text{area} \left( \triangle \right).
\]

Therefore

\[
\int_{0}^{3} x^2 \, dx = \frac{1}{2} \left[ 6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3 \right]
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= \frac{1}{2} \left[ 6 \cdot 9 \left( 1 - \frac{2}{3} \right) \right]
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\int_{0}^{3} x^2 \, dx = \frac{1}{2} \left[ 6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3 \right]
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= \frac{1}{2} \left[ 6 \cdot 9 \left( 1 - \frac{2}{3} \right) \right]
\]

\[
= 9.
\]
Having to estimate definite integrals is incredibly unsatisfying.
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**Theorem 1 (Fundamental Theorem of Calculus)**

If $F'(t)$ is a continuous function on $[a, b]$, then

$$
\int_{a}^{b} F'(t) \, dt = F(b) - F(a).
$$
Example

Let \( F(t) = \frac{1}{3} x^3 \).
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\int_0^3 x^2 \, dx = \int_0^3 F'(x) \, dx = F(3) - F(0)
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Remark 1
Essentially, this says that the area between the derivative of \( F \) and the \( x \)-axis from \( a \) to \( b \) is just the total change in \( F \) on the interval \([a, b]\).
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Remark 1

Essentially, this says that the area between the derivative of $F$ and the $x$-axis from $a$ and $b$ is just the total change in $F$ on the interval $[a, b]$. 
Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by
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Given the marginal cost $C'(q)$ and fixed costs:
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- Total variable cost to produce $b$ units:
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